Chapter 2

Solutions to Selected Exercises

Section 2.1

2. For all x, for all y, $x + y = y + x$.

3. An *isosceles trapezoid* is a trapezoid with equal legs.

5. The medians of any triangle intersect at a single point.

6. If $0 < x < 1$ and $\varepsilon > 0$, there exists a positive integer n satisfying $x^n < \varepsilon$.

8. Let m and n be odd integers. Then there exist k_1 and k_2 such that $m = 2k_1+1$ and $n = 2k_2+1$. Now
$$m + n = (2k_1 + 1) + (2k_2 + 1) = 2(k_1 + k_2 + 1).$$
Therefore, $m + n$ is even.

9. Let m and n be even integers. Then there exist k_1 and k_2 such that $m = 2k_1$ and $n = 2k_2$. Now
$$mn = (2k_1)(2k_2) = 2(2k_1k_2).$$
Therefore, mn is even.

11. Let m be an odd integer and n be an even integer. Then there exist k_1 and k_2 such that $m = 2k_1 + 1$ and $n = 2k_2$. Now
$$mn = (2k_1 + 1)(2k_2) = 2(2k_1k_2 + k_2).$$
Therefore, mn is even.

12. Let m and n be integers such that m and $m+n$ are even. Then there exist k_1 and k_2 such that $m = 2k_1$ and $m + n = 2k_2$. Now
$$n = (m + n) - m = 2k_2 - 2k_1 = 2(k_2 - k_1).$$
Therefore, n is even.

14. Let x and y be rational numbers. Then there exist integers m_1, n_1, m_2, n_2 such that $x = m_1/n_1$ and $y = m_2/n_2$. Now $xy = (m_1m_2)/(n_1n_2)$. Therefore xy is rational.

15. Let x be a nonzero rational number. Then there exist integers $m \neq 0$ and $n \neq 0$ such that $x = m/n$. Now $1/x = n/m$. Therefore $1/x$ is rational.

17. $$\begin{aligned} x \cdot 0 + 0 &= x \cdot 0 && \text{because } b + 0 = b \text{ for all real numbers } b \\ &= x \cdot (0 + 0) && \text{because } b + 0 = b \text{ for all real numbers } b \\ &= x \cdot 0 + x \cdot 0 && \text{because } a(b + c) = ab + ac \text{ for all real numbers } a, b, c \end{aligned}$$

    Taking $a = c = x \cdot 0$ and $b = 0$, the preceding equation becomes $a + b = a + c$; therefore, $0 = b = c = x \cdot 0$.

18. We must have $X = Y$. To prove this, suppose that $x \in X$. Since Y is nonempty, choose $y \in Y$. Then $(x, y) \in X \times Y$. Since $X \times Y = Y \times X$, $(x, y) \in Y \times X$. Therefore $x \in Y$. Similarly, if $x \in Y$, then $x \in X$. Thus $X = Y$.

20. Let $x \in X$. Then $x \in X \cup Y$. Therefore $X \subseteq X \cup Y$.

21. Let $x \in X \cup Z$. Then $x \in X$ or $x \in Z$. If $x \in X$, since $X \subseteq Y$, $x \in Y$. Therefore $x \in Y \cup Z$. If $x \in Z$, then $x \in Y \cup Z$. In either case, $x \in Y \cup Z$. Therefore $X \cup Z \subseteq Y \cup Z$.

23. Let $x \in Z - Y$. Then $x \in Z$ and $x \notin Y$. Now x cannot be in X, for if $x \in X$, since $X \subseteq Y$, then $x \in Y$, which is not the case. Since $x \in Z$ and $x \notin X$, $x \in Z - X$. Therefore $Z - Y \subseteq Z - X$.

24. Let $x \in Y - (Y - X)$. Then $x \in Y$ and $x \notin Y - X$. Since $x \in Y$, we must have $x \in X$ (if $x \notin X$, we would have $x \in Y - X$). Therefore $Y - (Y - X) \subseteq X$.

    Now let $x \in X$. Then $x \notin Y - X$. Since $X \subseteq Y$, $x \in Y$. Thus $x \in Y - (Y - X)$. Therefore $X \subseteq Y - (Y - X)$. We have shown that $Y - (Y - X) = X$.

26. Let $Z \in \mathcal{P}(X) \cup \mathcal{P}(Y)$. Then $Z \in \mathcal{P}(X)$ or $Z \in \mathcal{P}(Y)$. If $Z \in \mathcal{P}(X)$, then Z is a subset of X and, thus, Z is also a subset of $X \cup Y$. Therefore $Z \in \mathcal{P}(X \cup Y)$. Similarly, if $Z \in \mathcal{P}(Y)$, $Z \in \mathcal{P}(X \cup Y)$. In either case, $Z \in \mathcal{P}(X \cup Y)$. Therefore $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$.

27. Let $Z \in \mathcal{P}(X \cap Y)$. Then Z is a subset of $X \cap Y$. Therefore Z is a subset of X and a subset of Y. Thus $Z \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. We have proved that $\mathcal{P}(X \cap Y) \subseteq \mathcal{P}(X) \cap \mathcal{P}(Y)$.

    Let $Z \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. Then $Z \in \mathcal{P}(X)$ and $Z \in \mathcal{P}(Y)$. Since $Z \in \mathcal{P}(X)$, Z is a subset of X. Since $Z \in \mathcal{P}(Y)$, Z is a subset of Y. Since Z is a subset of X and Y, Z is a subset of $X \cap Y$. Thus $Z \in \mathcal{P}(X \cap Y)$. Therefore $\mathcal{P}(X) \cap \mathcal{P}(Y) \subseteq \mathcal{P}(X \cap Y)$. It follows that $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$.

29. Let $X = \{a\}$ and $Y = \{b\}$. Then

    $$\mathcal{P}(X) = \{\emptyset, \{a\}\}, \qquad \mathcal{P}(Y) = \{\emptyset, \{b\}\},$$

    so

    $$\mathcal{P}(X) \cup \mathcal{P}(Y) = \{\emptyset, \{a\}, \{b\}\}.$$

    Since $X \cup Y = \{a, b\}$,

    $$\mathcal{P}(X \cup Y) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

    Now $\{a, b\} \in \mathcal{P}(X \cup Y)$, but $\{a, b\} \notin \mathcal{P}(X) \cup \mathcal{P}(Y)$. Therefore $\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$ is false in general.

30.
$$
\begin{aligned}
(X \cap Y) - (X \cap Z) &= (X \cap Y) \cap \overline{(X \cap Z)} && [A - B = A \cap \overline{B}] \\
&= (X \cap Y) \cap (\overline{X} \cup \overline{Z}) && \text{[De Morgan's law; Theorem 1.1.21, part (k)]} \\
&= ((X \cap Y) \cap \overline{X}) \cup ((X \cap Y) \cap \overline{Z}) && \text{[Distributive law; Theorem 1.1.21, part (c)]} \\
&= ((Y \cap X) \cap \overline{X}) \cup ((X \cap Y) \cap \overline{Z}) && \text{[Commutative law; Theorem 1.1.21, part (b)]} \\
&= (Y \cap (X \cap \overline{X})) \cup (X \cap (Y \cap \overline{Z})) && \text{[Associative law; Theorem 1.1.21, part (a)]} \\
&= (Y \cap \emptyset) \cup (X \cap (Y \cap \overline{Z})) && \text{[Complement law; Theorem 1.1.21, part (e)]} \\
&= \emptyset \cup (X \cap (Y \cap \overline{Z})) && \text{[Bound law; Theorem 1.1.21, part (g)]} \\
&= (X \cap (Y \cap \overline{Z})) \cup \emptyset && \text{[Commutative law; Theorem 1.1.21, part (b)]} \\
&= X \cap (Y \cap \overline{Z}) && \text{[Identity law; Theorem 1.1.21, part (d)]} \\
&= X \cap (Y - Z) && [A - B = A \cap \overline{B}]
\end{aligned}
$$

32. False. Let $X = \{a\}$ and $Y = Z = \{b\}$. Then

$$X \cup (Y - Z) = \{a\}, \qquad (X \cup Y) - (X \cup Z) = \emptyset.$$

33. True. $\overline{Y - X} = \overline{Y \cap \overline{X}} = \overline{Y} \cup \overline{\overline{X}} = \overline{Y} \cup X = X \cup \overline{Y}$.

35. False. Let $X = \{a\}$ and $Y = Z = \{b\}$. Then

$$X - (Y \cup Z) = \{a\}, \qquad (X - Y) \cup Z = \{a, b\}.$$

36. False. Let $X = \{a\}$, $Y = \{b\}$, and $U = \{a, b\}$. Then

$$\overline{X - Y} = \{b\}, \qquad \overline{Y - X} = \{a\}.$$

38. True. Let $x \in (X \cap Y) \cup (Y - X)$. Now either $x \in X \cap Y$ or $x \in Y - X$. In either case, $x \in Y$. Therefore $(X \cap Y) \cup (Y - X) \subseteq Y$.

Now suppose that $x \in Y$. Either $x \in X$ or $x \notin X$. If $x \in X$, then $x \in X \cap Y$. Thus $x \in (X \cap Y) \cup (Y - X)$. If $x \notin X$, then $x \in Y - X$. Again $x \in (X \cap Y) \cup (Y - X)$. Thus $Y \subseteq (X \cap Y) \cup (Y - X)$. Therefore $(X \cap Y) \cup (Y - X) = Y$.

39. True. Let $a \in X \times (Y \cup Z)$. Then $a = (x, y)$ where $x \in X$ and $y \in Y \cup Z$. Now $y \in Y$ or $y \in Z$. If $y \in Y$, then $a = (x, y) \in X \times Y$. Thus $a \in (X \times Y) \cup (X \times Z)$. If $y \in Z$, then $a = (x, y) \in X \times Z$. Again $a \in (X \times Y) \cup (X \times Z)$. Therefore $X \times (Y \cup Z) \subseteq (X \times Y) \cup (X \times Z)$.

Now suppose that $a \in (X \times Y) \cup (X \times Z)$. Then either $a \in X \times Y$ or $a \in X \times Z$. If $a \in X \times Y$, then $a = (x, y)$ where $x \in X$ and $y \in Y$. In particular, $y \in Y \cup Z$. Thus $a = (x, y) \in X \times (Y \cup Z)$. If $a \in X \times Z$, then $a = (x, z)$ where $x \in X$ and $z \in Z$. In particular, $z \in Y \cup Z$. Thus $a = (x, z) \in X \times (Y \cup Z)$. Therefore $(X \times Y) \cup (X \times Z) \subseteq X \times (Y \cup Z)$. We have proved that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

41. True. Let $a \in X \times (Y - Z)$. Then $a = (x, y)$, where $x \in X$ and $y \in Y - Z$. Thus $y \in Y$ and $y \notin Z$ and, so, $(x, y) \in X \times Y$ and $(x, y) \notin X \times Z$. Therefore $a = (x, y) \in (X \times Y) - (X \times Z)$. We have shown that $X \times (Y - Z) \subseteq (X \times Y) - (X \times Z)$.

    Now suppose that $a \in (X \times Y) - (X \times Z)$. Then $a \in X \times Y$ and $a \notin X \times Z$. Thus $a = (x, y)$, where $x \in X$, $y \in Y$, and $y \notin Z$. Therefore $a = (x, y) \in X \times (Y - Z)$. We have shown that $(X \times Y) - (X \times Z) \subseteq X \times (Y - Z)$. It follows that $X \times (Y - Z) = (X \times Y) - (X \times Z)$.

42. False. Take $X = \{1, 2\}$, $Y = \{1\}$, $Z = \{2\}$. Then

$$Y \times Z = \{(1, 2)\}, \quad X - Y = \{2\}, \quad X - Z = \{1\}.$$

    Thus

$$X - (Y \times Z) = \{1, 2\} \quad \text{and} \quad (X - Y) \times (X - Z) = \{(2, 1)\}.$$

45–54. Argue as in the proof given in the book of the first associative law [Theorem 1.1.21, part (a)].

56. By definition

$$(A \triangle B) \triangle A = [(A \triangle B) \cup A] - [(A \triangle B) \cap A].$$

    Show that

$$(A \triangle B) \cup A = A \cup B \quad \text{and} \quad (A \triangle B) \cap A = A \cap \overline{B}.$$

    The statement then follows easily.

57. The statement is true. We first prove that $A \subseteq B$. Let $x \in A$.

    We divide the proof into two cases. First, we consider the case that $x \in C$. Then $x \notin A \triangle C$. Therefore $x \notin B \triangle C$. Thus $x \in B$ (since if $x \notin B$, then $x \in B \triangle C$).

    Next, we consider the case that $x \notin C$. Then $x \in A \triangle C$. Therefore $x \in B \triangle C$. Thus $x \in B$.

    In either case, $x \in B$, and so $A \subseteq B$. Similarly, $B \subseteq A$, and so $A = B$.

59. The statement is false. Let

$$A = \{1, 2\}, \quad B = \{2, 3\}, \quad C = \{1, 3\}.$$

    Since $B \cap C = \{3\}$,

$$A \triangle (B \cap C) = \{1, 2, 3\}.$$

    Now

$$A \triangle B = \{1, 3\} \quad \text{and} \quad A \triangle C = \{2, 3\},$$

    thus

$$(A \triangle B) \cap (A \triangle C) = \{3\}.$$

60. The statement is false. Let

$$A = \{1, 2\}, \quad B = \{2, 3\}, \quad C = \{1, 3\}.$$

    Since $B \triangle C = \{1, 2\}$,

$$A \cup (B \triangle C) = \{1, 2\}.$$

    Since $A \cup B = A \cup C = \{1, 2, 3\}$,

$$(A \cup B) \triangle (A \cup C) = \emptyset.$$

62. Yes, $\triangle$ is commutative:

$$A \triangle B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B \triangle A.$$

63. Yes, $\triangle$ is associative. We first prove that

$$(A \triangle B) \triangle C = (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap C). \tag{2.1}$$

[For the motivation of this formula, draw the Venn diagram of $(A \triangle B) \triangle C$.] By Exercise 55,

$$(A \triangle B) \triangle C = [(A \triangle B) - C] \cup [C - (A \triangle B)].$$

Again using Exercise 55 and the fact that $X - Y = X \cap \overline{Y}$, we have

$$(A \triangle B) - C = [(A - B) \cup (B - A)] - C = [(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}.$$

Using the definition of $\triangle$, the fact that $X - Y = X \cap \overline{Y}$, and De Morgan's laws, we have

$$\overline{A \triangle B} = \overline{(A \cup B) - (A \cap B)} = \overline{(A \cup B) \cap \overline{(A \cap B)}} = \overline{(A \cup B)} \cup \overline{\overline{(A \cap B)}} = (\overline{A} \cap \overline{B}) \cup (A \cap B).$$

Thus

$$C - (A \triangle B) = C \cap \overline{(A \triangle B)} = C \cap [(\overline{A} \cap \overline{B}) \cup (A \cap B)].$$

Combining the preceding equations and using Theorem 1.1.21, we obtain equation (2.1)

$$\begin{aligned}
(A \triangle B) \triangle C &= [(A \triangle B) - C] \cup [C - (A \triangle B)] \\
&= \{[(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}\} \cup \{C \cap [(\overline{A} \cap \overline{B}) \cup (A \cap B)]\} \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap C).
\end{aligned}$$

By Exercise 62, $\triangle$ is commutative. Thus

$$A \triangle (B \triangle C) = (B \triangle C) \triangle A.$$

We can obtain a formula for $(B \triangle C) \triangle A$ using equation (2.1) with A replaced by B, B replaced by C, and C replaced by A. However, noting that the right-hand side of equation (2.1) is symmetric in A, B, and C, we see that the two expressions

$$(A \triangle B) \triangle C \quad \text{and} \quad A \triangle (B \triangle C)$$

are equal. Therefore, $\triangle$ is associative.

Section 2.2

2. False; $x = \sqrt{2}$ is a counterexample.

3. We prove the contrapositive: If x is rational, then x^3 is rational.

   Suppose that x is rational. Then there exist integers p and q such that $x = p/q$. Now $x^3 = p^3/q^3$. Thus x^3 is rational.

5. Suppose, by way of contradiction, that $x < 1$ and $y < 1$ and $z < 1$. Adding these inequalities gives $x + y + z < 3$, which is a contradiction.

6. Suppose, by way of contradiction, that $x > \sqrt{2}$ and $y > \sqrt{2}$. Multiplying these inequalities gives $xy > 2$, which is a contradiction.

8. Suppose, by way of contradiction, that $x + y$ is rational. Since x and $x + y$ are rational, there exist integers p_1, p_2, q_1, q_2 such that $x = p_1/q_1$ and $x + y = p_2/q_2$. Now

$$y = (x + y) - x = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2}.$$

Therefore y is rational, which is a contradiction.

9. False; a counterexample is $x = 0, y = \sqrt{2}$.

11. Since the integers increase without bound, there exists $n \in \mathbf{Z}$ such that $\sqrt{2}/(b - a) < n$. Therefore $\sqrt{2}/n < b - a$. Choose $m \in \mathbf{Z}$ as large as possible satisfying $m\sqrt{2}/n \le a$. Then, by the choice of m, $a < (m + 1)\sqrt{2}/n$. Also

$$\frac{(m+1)\sqrt{2}}{n} = \frac{m\sqrt{2}}{n} + \frac{\sqrt{2}}{n} < a + (b - a) = b.$$

Therefore $x = (m + 1)\sqrt{2}/n$ is an irrational number satisfying $a < x < b$. (If $(m + 1)\sqrt{2}/n$ is rational, say $(m + 1)\sqrt{2}/n = p/q$ where p and q are integers, then $\sqrt{2} = np/[(m + 1)q]$ is rational, which is not the case.)

12. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we have found irrational numbers a and b (namely $a = b = \sqrt{2}$) such that a^b is rational. Suppose that $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Now $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$ is rational. We have found irrational numbers a and b such that a^b is rational.

This proof is nonconstructive since it does not show whether the desired pair is $a = b = \sqrt{2}$ or $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$.

14. Let $a = 2$ and $b = 1/2$. Then a and b are rational. Now $a^b = 2^{1/2} = \sqrt{2}$ is irrational. This proof is a constructive existence proof.

15. Suppose, by way of contradiction, that $x > y$. Let $\varepsilon = (x - y)/2$. Then

$$y + \varepsilon = y + \frac{x - y}{2} = \frac{x + y}{2} < \frac{x + x}{2} = x,$$

which is a contradiction.

17. Suppose, by way of contradiction, that $X \times \emptyset$ is not empty. Then there exists $(x, y) \in X \times \emptyset$. Now $y \in \emptyset$, which is a contradiction.

18. Suppose that every box contains less than 12 balls. Then each box contains at most 11 balls and the maximum number of balls contained by the nine boxes is $9 \cdot 11 = 99$. Contradiction.

20. Let i be the greatest integer for which s_i is positive. Since s_1 is positive and the set of indexes $1, 2, \ldots, n$ is finite, such an i exists. Since s_n is negative, $i < n$. Now s_{i+1} is equal to either s_i+1 or $s_i - 1$. If $s_{i+1} = s_i + 1$, then s_{i+1} is a positive integer (since s_i is a positive integer). This contradicts the fact that i is the *greatest* integer for which s_i is positive. Therefore, $s_{i+1} = s_i - 1$. Again, if $s_i - 1$ is a positive integer, we have a contradiction. Therefore, $s_{i+1} = s_i - 1 = 0$.

21. For $n = 3$, we have $n^2 > 2^n$.

23. The statement is false. Let $s_1 = s_2 = 3$. Then $A = 3$. For no i do we have $s_i > A$. The proof is by counterexample.

24. The statement is true and we prove it using proof by contradiction. Suppose that for every j, $s_j \le A$. Since $s_j \le A$ for all j and $s_i < A$,

$$s_1 + \cdots + s_i + \cdots + s_n < A + \cdots + A + \cdots + A = nA.$$

Dividing by n, we obtain

$$\frac{s_1 + \cdots + s_n}{n} < A,$$

which is a contradiction.

26. Since $s_i \ne s_j$, either $s_i \ne A$ or $s_j \ne A$. By changing the notation, if necessary, we may assume that $s_i \ne A$. Either $s_i < A$ or $s_i > A$. If $s_i > A$, the proof is complete; so assume that $s_i < A$. We show that there exists k such that $s_k > A$. Suppose, by way of contradiction, that $s_m \le A$ for all m, that is,

$$\begin{array}{rcl} s_1 & \le & A \\ s_2 & \le & A \\ & \vdots & \\ s_n & \le & A. \end{array}$$

Adding these inequalities yields

$$s_1 + s_2 + \cdots + s_i + \cdots + s_n < nA$$

since $s_i < A$. Dividing by n gives

$$\frac{s_1 + s_2 + \cdots + s_n}{n} < A,$$

which is a contradiction. Therefore there exists k such that $s_k > A$.

28. If m and n are positive integers and $m > 3$, then $m^3 + 2n^2 > 36$. If m and n are positive integers and $n > 4$, then $m^3 + 2n^2 > 36$. Thus it suffices to consider the cases $1 \le m \le 3$ and $1 \le n \le 4$. The following table, which shows that values of $m^3 + 2n^2$, shows that there is no solution to $m^3 + 2n^2 = 36$:

		$m=1$	$m=2$	$m=3$
n	1	3	10	29
	2	9	16	35
	3	19	26	45
	4	33	40	59

29. Notice that $2m^2 + 4n^2 - 1$ is odd and $2(m+n)$ is even. Therefore $2m^2 + 4n^2 - 1 \neq 2(m+n)$ for all positive integers m and n.

31. We consider two cases: n is even, n is odd. First suppose that n is even. By Exercise 9, Section 2.1, the product of even integers is even. Therefore $n^2 = n \cdot n$ is even. Again by Exercise 9, Section 2.1, $n^3 = n^2 \cdot n$ is even. By Exercise 7, Section 2.1, the sum of even integers is even. Therefore $n^3 + n$ is even.

    Now suppose that n is odd. By Exercise 10, Section 2.1, the product of odd integers is odd. Therefore $n^2 = n \cdot n$ is odd. Again by Exercise 10, Section 2.1, $n^3 = n^2 \cdot n$ is odd. By Exercise 8, Section 2.1, the sum of odd integers is even. Therefore $n^3 + n$ is even. In either case, $n^3 + n$ is even.

33. First, note that from Exercise 32, for all x,

$$|-x| = |(-1)x| = |-1||x| = |x|.$$

Example 2.2.6 states that for all x, $x \leq |x|$. Using these results, we consider two cases: $x+y \geq 0$ and $x+y < 0$. If $x+y \geq 0$, we have

$$|x+y| = x+y \leq |x| + |y|.$$

If $x+y < 0$, we have

$$|x+y| = -(x+y) = -x + -y \leq |-x| + |-y| = |x| + |y|.$$

35. Suppose that $xy > 0$. Then either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. If $x > 0$ and $y > 0$,

$$\operatorname{sgn}(xy) = 1 = 1 \cdot 1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

If $x < 0$ and $y < 0$,

$$\operatorname{sgn}(xy) = 1 = -1 \cdot -1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

Next, suppose that $xy = 0$. Then either $x = 0$ or $y = 0$. Thus either $\operatorname{sgn}(x) = 0$ or $\operatorname{sgn}(y) = 0$. In either case, $\operatorname{sgn}(x)\operatorname{sgn}(y) = 0$. Therefore

$$\operatorname{sgn}(xy) = 0 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

Finally, suppose that $xy < 0$. Then either $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$. If $x > 0$ and $y < 0$,

$$\operatorname{sgn}(xy) = -1 = 1 \cdot -1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

If $x < 0$ and $y > 0$,

$$\operatorname{sgn}(xy) = -1 = -1 \cdot 1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

36. $|xy| = \mathrm{sgn}(xy)xy = \mathrm{sgn}(x)\mathrm{sgn}(y)xy = [\mathrm{sgn}(x)x][\mathrm{sgn}(y)y] = |x||y|$

38. Suppose that $x \geq y$. Then

$$\max\{x, y\} = x \qquad \text{and} \qquad |x - y| = x - y.$$

Thus

$$\max\{x, y\} = x = \frac{2x}{2} = \frac{x + y + x - y}{2} = \frac{x + y + |x - y|}{2}.$$

The other case is $x < y$. Then

$$\max\{x, y\} = y \qquad \text{and} \qquad |x - y| = y - x.$$

Thus

$$\max\{x, y\} = y = \frac{2y}{2} = \frac{x + y + y - x}{2} = \frac{x + y + |x - y|}{2}.$$

39. Suppose that $x \geq y$. Then

$$\min\{x, y\} = y \qquad \text{and} \qquad |x - y| = x - y.$$

Thus

$$\min\{x, y\} = y = \frac{2y}{2} = \frac{x + y - (x - y)}{2} = \frac{x + y - |x - y|}{2}.$$

The other case is $x < y$. Then

$$\min\{x, y\} = x \qquad \text{and} \qquad |x - y| = y - x.$$

Thus

$$\min\{x, y\} = x = \frac{2x}{2} = \frac{x + y - (y - x)}{2} = \frac{x + y - |x - y|}{2}.$$

40.
$$\begin{aligned}\max\{x, y\} + \min\{x, y\} &= \frac{x + y + |x - y|}{2} + \frac{x + y - |x - y|}{2} \\ &= \frac{x + y + |x - y| + x + y - |x - y|}{2} \\ &= \frac{2x + 2y}{2} = x + y.\end{aligned}$$

42. Suppose that n is odd. Then $n = 2k + 1$. Now $n + 2 = (2k + 1) + 2 = 2(k + 1) + 1$ is odd.

Now suppose that $n + 2$ is odd. Then $n + 2 = 2k + 1$. Now $n = (2k + 1) - 2 = 2(k - 1) + 1$ is odd.

Therefore n is odd if and only if $n + 2$ is even.

44. Suppose that $A \subseteq C$ and $B \subseteq C$. Let $x \in A \cup B$. Then either $x \in A$ or $x \in B$. If $x \in A$, since $A \subseteq C$, $x \in C$. If $x \in B$, since $B \subseteq C$, $x \in C$. In either case, $x \in C$. Therefore $A \cup B \subseteq C$.

Now suppose that $A \cup B \subseteq C$. Let $x \in A$. Then $x \in A \cup B$. Since $A \cup B \subseteq C$, $x \in C$. Therefore $A \subseteq C$. Let $x \in B$. Then $x \in A \cup B$. Since $A \cup B \subseteq C$, $x \in C$. Therefore $B \subseteq C$. We conclude that $A \subseteq C$ and $B \subseteq C$. It follows that $A \subseteq C$ and $B \subseteq C$ if and only if $A \cup B \subseteq C$.

45. Suppose that $C \subseteq A$ and $C \subseteq B$. Let $x \in C$. Since $C \subseteq A$, $x \in A$. Since $C \subseteq B$, $x \in B$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Therefore $C \subseteq A \cap B$.

    Now suppose that $C \subseteq A \cap B$. Let $x \in C$. Then $x \in A \cap B$. In particular, $x \in A$. Therefore $C \subseteq A$. Again let $x \in C$. Then $x \in A \cap B$. In particular, $x \in B$. Therefore $C \subseteq B$. Thus $C \subseteq A$ and $C \subseteq B$. It follows that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$.

48. [(a) $\rightarrow$ (b)] We assume that $A \cap B = \emptyset$ and prove that $B \subseteq \overline{A}$. Let $x \in B$. If $x \in A$, we obtain the contradiction $A \cap B \neq \emptyset$. Thus $x \notin A$. Hence $x \in \overline{A}$. Therefore $B \subseteq \overline{A}$.

    [(b) $\rightarrow$ (c)] We assume that $B \subseteq \overline{A}$ and prove that $A \triangle B = A \cup B$.

    Let $x \in A \triangle B$. By definition, $A \triangle B = (A \cup B) - (A \cap B)$, thus $x \in A \cup B$. Therefore $A \triangle B \subseteq A \cup B$.

    Let $x \in A \cup B$. We first prove that $x \notin A \cap B$. Suppose, by way of contradiction, that $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $B \subseteq \overline{A}$, $x \in \overline{A}$, which implies that $x \notin A$. We have the desired contradiction. Therefore $x \notin A \cap B$. Now $x \in (A \cup B) - (A \cap B) = A \triangle B$. Therefore $A \cup B \subseteq A \triangle B$. It follows that $A \triangle B = A \cup B$.

    [(c) $\rightarrow$ (a)] We assume that $A \triangle B = A \cup B$ and prove that $A \cap B = \emptyset$. Suppose, by way of contradiction, that $A \cap B$ is not empty. Then there exists $x \in A \cap B$. Then $x \in A \cup B$. This implies that $x \notin A \triangle B$. Since $x \in A \cup B$, $A \triangle B \neq A \cup B$, which is a contradiction. Therefore $A \cap B = \emptyset$.

49. [(a) $\rightarrow$ (b)] We assume that $A \cup B = U$ and prove that $\overline{A} \cap \overline{B} = \emptyset$. Taking the complement of both sides of the equation $A \cup B = U$ and using De Morgan's law and the 0/1 law (Theorem 1.1.21), we obtain

$$\overline{A} \cap \overline{B} = \overline{A \cup B} = \overline{U} = \emptyset.$$

    [(b) $\rightarrow$ (c)] We assume that $\overline{A} \cap \overline{B} = \emptyset$ and prove that $\overline{A} \subseteq B$. Replace A by $\overline{B}$ and B by $\overline{A}$ in Exercise 48(a) to obtain $\overline{B} \cap \overline{A} = \emptyset$. Since Exercise 48(a) is equivalent to Exercise 48(b), we obtain $\overline{A} \subseteq \overline{\overline{B}}$ or $\overline{A} \subseteq B$.

    [(c) $\rightarrow$ (a)] We assume that $\overline{A} \subseteq B$ and prove that $A \cup B = U$. Since U is a universal set, we automatically have $A \cup B \subseteq U$.

    Let $x \in U$. If $x \in A$, then $x \in A \cup B$. If $x \notin A$, then $x \in \overline{A}$. Since $\overline{A} \subseteq B$, $x \in B$. Again $x \in A \cup B$. Therefore $U \subseteq A \cup B$. It follows that $A \cup B = U$.

Problem-Solving Corner: Proofs

1. The least upper bound of a nonempty finite set of real numbers is the maximum number in the set.

2. Call the given set X. We prove that the least upper bound of X is 1. Since

$$1 - \frac{1}{n} < 1$$

for all positive integers n, 1 is an upper bound of X. Let a be an upper bound for X. Suppose, by way of contradiction, that $a < 1$. Since the integers are unbounded, there exists a positive integer k such that

$$\frac{1}{1-a} < k.$$

Multiplying by $(1-a)/k$ gives

$$\frac{1}{k} < 1-a,$$

which, in turn, is equivalent to

$$a < 1 - \frac{1}{k}.$$

This contradicts the fact that a is an upper bound of X. Thus $1 \le a$ and 1 is the least upper bound of X.

3. Let b be the least upper bound of Y. If $x \in X$, then $x \in Y$ and $x \le b$. Thus b is an upper bound of X. If a is the least upper bound of X, $a \le b$.

4. 0

5. Let $Z = \{x + y \mid x \in X \text{ and } y \in Y\}$ and let $z \in Z$. Then $z = x + y$ for some $x \in X, y \in Y$. Now $z = x + y \le a + b$. Therefore Z is bounded above by $a + b$.

   Let c be an upper bound of Z. Suppose, by way of contradiction, that $c < a+b$. Let $\varepsilon = a+b-c$. Now $a - \varepsilon/2$ is not an upper bound of X so there exists $x \in X$ such that

$$a - \frac{\varepsilon}{2} < x.$$

   Similarly, there exists $y \in Y$ such that

$$b - \frac{\varepsilon}{2} < y.$$

   Adding the previous inequalities gives

$$c = a + b - \varepsilon < x + y,$$

   which contradicts the fact that c is an upper bound of Z. Therefore $c \ge a + b$ and $a + b$ is the least upper bound of Z.

6. Since a is a greatest lower bound for X and b is a lower bound for X, $b \le a$. Since b is a greatest lower bound for X and a is a lower bound for X, $a \le b$. Therefore $a = b$.

7. Let X be a nonempty set of real numbers bounded below. Let Y be the set of lower bounds of X. The set Y is nonempty since X is bounded below. Let x be an element of X. For every $y \in Y$, we have $y \le x$ since y is a lower bound of X. Therefore Y is bounded above by x. Thus Y is has a least upper bound, say a.

   Next we show that a is a lower bound of X. Suppose, by way of contradiction, that a is not a lower bound of X. Then there exists $x \in X$ such that $x < a$. Then x is not an upper bound of Y. Therefore there exists $y \in Y$ such that $x < y$. But this contradicts the fact that y is a lower bound of X. Therefore a is a lower bound of X.

   Finally, we show that a is the greatest lower bound of X. Let b be a lower bound of X. Then $b \in Y$. Since a is an upper bound of Y, $b \le a$. Therefore a is the greatest lower bound of X.

8. Since $a + \varepsilon > a$, $a + \varepsilon$ is not a lower bound of X. Therefore there exists $x \in X$ such that $a + \varepsilon > x$. Since a is a lower bound of X, $x \geq a$.

9. Let tX denote the set
$$\{tx \mid x \in X\}.$$
We must prove that
   (a) $z \geq ta$ for every $z \in tX$ (i.e., ta is an lower bound for tX),
   (b) if b is an lower bound for tX, then $b \leq ta$ (i.e., ta is the greatest lower bound for tX).

   We first prove part (a). Let $z \in tX$. Then $z = tx$ for some $x \in X$. Since a is an upper bound for X, $x \leq a$. Multiplying by t and noting that $t < 0$, we have $z = tx \geq ta$. Therefore, $z \geq ta$ for every $z \in tX$ and the proof of part (a) is complete.

   Next we prove part (b). Let b be a lower bound for tX. Then $tx \geq b$ for every $x \in X$. Dividing by t and noting that $t < 0$, we have $x \leq b/t$ for every $x \in X$. Therefore b/t is an upper bound for X. Since a is the least upper bound for X, $b/t \geq a$. Multiplying by t and noting again that $t < 0$, we have $b \leq ta$. Therefore ta is the greatest lower bound for tX. The proof is complete.

Section 2.3

3.

1.	$\neg p \vee r$	
2.	$\neg r \vee q$	
3.	p	
4.	$\neg p \vee q$	from 1,2
5.	q	from 3,4

4.

1.	$\neg p \vee t$	
2.	$\neg q \vee s$	
3.	$\neg r \vee s$	
4.	$\neg r \vee t$	
5.	$p \vee q \vee r \vee u$	
6.	$t \vee q \vee r \vee u$	from 1,5
7.	$s \vee t \vee r \vee u$	from 2,6
8.	$s \vee t \vee u$	from 3,7

6. $(p \leftrightarrow r) \equiv (p \to r)(r \to p) \equiv (\neg p \vee r)(\neg r \vee p)$

1.	$\neg p \vee r$	
2.	$\neg r \vee p$	
3.	r	
4.	p	from 2,3

8.

1.	$a \vee \neg b$	
2.	$a \vee c$	
3.	$\neg a$	
4.	$\neg d$	
5.	b	negated conclusion
6.	$\neg b$	from 1,3

Now 5 and 6 combine to give a contradiction.

Section 2.4

In some of these solutions, the Basis Steps are omitted.

2. $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2)$
$$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

3. $1(1!) + 2(2!) + \cdots + n(n!) + (n+1)(n+1)!$
$$= (n+1)! - 1 + (n+1)(n+1)! = (n+2)! - 1$$

5. $1^2 - 2^2 + \cdots + (-1)^{n+1}n^2 + (-1)^{n+2}(n+1)^2$
$$= \frac{(-1)^{n+1}n(n+1)}{2} + (-1)^{n+2}(n+1)^2 = \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

6. $1^3 + 2^3 + \cdots + n^3 + (n+1)^3$
$$= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$$

8. $\dfrac{1}{2 \cdot 4} + \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 6} + \cdots + \dfrac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n+2)} + \dfrac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n+2)(2n+4)}$
$$= \frac{1}{2} - \frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n+2)} + \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n+2)(2n+4)}$$
$$= \frac{1}{2} - \frac{1 \cdot 3 \cdots (2n+3)}{2 \cdot 4 \cdots (2n+4)}$$

9. $\dfrac{1}{2^2-1} + \dfrac{1}{3^2-1} + \cdots + \dfrac{1}{(n+1)^2-1} + \dfrac{1}{(n+2)^2-1}$
$$= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} + \frac{1}{(n+2)^2-1}$$
$$= \frac{3}{4} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

11. The solution is similar to that for Exercise 10, which is given in the book.

13. First note that
$$\frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n)(2n+2)} \le \frac{1}{\sqrt{n+1}} \frac{2n+1}{2n+2}.$$

The proof will be complete if we can show that
$$\frac{2n+1}{(2n+2)\sqrt{n+1}} \le \frac{1}{\sqrt{n+2}}.$$

This last inequality is successively equivalent to

$$\begin{aligned}
\left(\frac{n+2}{n+1}\right)^{1/2} &\leq \frac{2n+2}{2n+1} \\
\frac{n+2}{n+1} &\leq \frac{4(n+1)^2}{(2n+1)^2} \\
(n+2)(2n+1)^2 &\leq 4(n+1)^3 \\
4n^3+12n^2+9n+2 &\leq 4n^3+12n^2+12n+4 \\
-2 &\leq 3n.
\end{aligned}$$

This last inequality is true for all $n \geq 1$.

14. $2(n+1)+1=(2n+1)+2 \leq 2^n+2 \leq 2^n+2^n=2^{n+1}$

16. By the inductive assumption,

$$(a_1 \cdots a_{2^n})^{1/2^n} \leq \frac{a_1+\cdots+a_{2^n}}{2^n} \tag{2.2}$$

$$(a_{2^n+1} \cdots a_{2^{n+1}})^{1/2^n} \leq \frac{a_{2^n+1}+\cdots+a_{2^{n+1}}}{2^n}. \tag{2.3}$$

Let

$$A=\frac{a_1+\cdots+a_{2^n}}{2^n} \quad \text{and} \quad B=\frac{a_{2^n+1}+\cdots+a_{2^{n+1}}}{2^n}.$$

Multiplying inequalities (2.2) and (2.3), we have

$$(a_1 \cdots a_{2^{n+1}})^{1/2^n} \leq AB. \tag{2.4}$$

Applying the Basis Step to the numbers A and B, we have

$$(AB)^{1/2} \leq \frac{A+B}{2}$$

or, equivalently,

$$AB \leq \left[\frac{a_1+\cdots+a_{2^{n+1}}}{2^{n+1}}\right]^2. \tag{2.5}$$

Combining inequalities (2.4) and (2.5), we have

$$(a_1 \cdots a_{2^{n+1}})^{1/2^n} \leq \left[\frac{a_1+\cdots+a_{2^{n+1}}}{2^{n+1}}\right]^2.$$

Taking the square root of both sides of the last inequality gives the desired result.

17.
$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geq (1+nx)(1+x) \\
&= 1+nx+x+nx^2 \\
&\geq 1+(n+1)x
\end{aligned}$$

19. If we sum the terms in the diagonal direction, we obtain one r, two r^2's, three r^3's, and so on; that is, we obtain the sum
$$1 \cdot r^1 + 2 \cdot r^2 + \cdots + nr^n.$$
Multiplying the inequality of Exercise 18 by r yields
$$r^1 + r^2 + \cdots + r^{n+1} < \frac{r}{1-r} \qquad \text{for all } n \geq 0. \tag{2.6}$$
Thus, the sum of the entries in the first column is less than $r/(1-r)$. Similarly, the sum of the entries in the second column is less than $r^2/(1-r)$, and so on. It follows from the preceding discussion that
$$1 \cdot r^1 + 2 \cdot r^2 + \cdots + nr^n < \frac{1}{1-r}(r^1 + r^2 + \cdots + r^n).$$
Using inequality (2.6), we obtain the desired result
$$1 \cdot r^1 + 2 \cdot r^2 + \cdots + nr^n < \frac{1}{1-r}(r^1 + r^2 + \cdots + r^n) < \left(\frac{1}{1-r}\right)\left(\frac{r}{1-r}\right) = \frac{r}{(1-r)^2}.$$

20. Take $r = 1/2$ in Exercise 19.

22. Assume that $11^n - 6$ is divisible by 5.
$$11^{n+1} - 6 = 11^n \cdot 11 - 6 = 11^n(10+1) - 6 = 10 \cdot 11^n + 11^n - 6,$$
which is divisible by 5.

23. Suppose that 4 divides $6 \cdot 7^n - 2 \cdot 3^n$. Now
$$\begin{aligned} 6 \cdot 7^{n+1} - 2 \cdot 3^{n+1} &= 7 \cdot 6 \cdot 7^n - 3 \cdot 2 \cdot 3^n \\ &= 6 \cdot 7^n - 2 \cdot 3^n + 6 \cdot 6 \cdot 7^n - 2 \cdot 2 \cdot 3^n \\ &= 6 \cdot 7^n - 2 \cdot 3^n + 36 \cdot 7^n - 4 \cdot 3^n. \end{aligned}$$
Since 4 divides $6 \cdot 7^n - 2 \cdot 3^n$, $36 \cdot 7^n$, and $-4 \cdot 3^n$, it divides their sum, which is $6 \cdot 7^{n+1} - 2 \cdot 3^{n+1}$.

25. We prove part (a) only. The Basis Step is immediate.
Assume that
$$X \cap (X_1 \cup X_2 \cup \cdots \cup X_n) = (X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_n).$$
We must prove that
$$X \cap (X_1 \cup X_2 \cup \cdots \cup X_n \cup X_{n+1}) = (X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_n) \cup (X \cap X_{n+1}).$$
Let $Y = X_n \cup X_{n+1}$. By the inductive assumption,
$$X \cap (X_1 \cup X_2 \cup \cdots \cup X_{n-1} \cup Y) = (X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_{n-1}) \cup (X \cap Y).$$
By the associative law,
$$X \cap (X_1 \cup X_2 \cup \cdots \cup X_{n-1} \cup Y) = X \cap (X_1 \cup X_2 \cup \cdots \cup X_n \cup X_{n+1}).$$

By the distributive law,

$$X \cap Y = X \cap (X_n \cup X_{n+1}) = (X \cap X_n) \cup (X \cap X_{n+1}).$$

Therefore

$$(X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_{n-1}) \cup (X \cap Y) = (X \cap X_1) \cup (X \cap X_2) \cup \cdots \cup (X \cap X_n) \cup (X \cap X_{n+1}),$$

and the Inductive Step is complete.

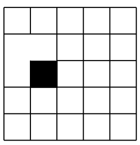
28. $\dfrac{n}{n+1}$

30. We use induction on n, the number of lines, to prove the result. If there is one line, the result is certainly true. Suppose that there are $n > 1$ lines. Remove one line. By the inductive hypothesis, the regions that result may be colored red and green so that no two regions that share an edge are the same color. Now restore the removed line. The regions above (or, if the line is vertical, to the left of) the restored line are colored red and green so that no two regions that share an edge are the same color, and the regions below (or, if the line is vertical, to the right of) the restored line are also colored red and green so that no two regions that share an edge are the same color. Now reverse the color of every region below (or, if the line is vertical, to the right of) the restored line. The regions below (or, if the line is vertical, to the right of) the restored line are still colored red and green so that no two regions that share an edge are the same color. Since the colors below the restored line have been reversed, regions that share an edge that is part of the restored line do not have the same color. Therefore the regions may be colored red and green so that no two regions that share an edge are the same color, and the inductive proof is complete.

31. The proof is by induction on the number n of zeros with the Basis Step, as usual, omitted.

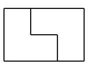
    Suppose that the result is true for n zeros, and we are given $n+1$ zeros and $n+1$ ones distributed around a circle. Find a zero followed, in clockwise order, by a one. Temporarily remove these two numbers. By the inductive assumption, it is possible to start at some number and proceed around the circle to the original starting position in such a way that, at any point during the cycle, one has seen at least as many zeros as ones. Notice that this last statement remains true if we restore the removed zero and one.

33. A tromino can cover the square to the left of the missing square as shown

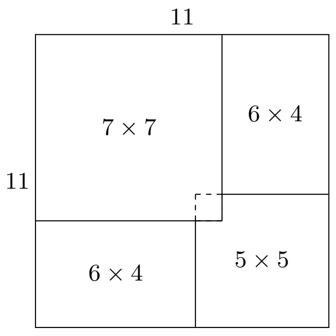
    

    or in a symmetric fashion by reversing "up" and "down." In the first case, it is impossible to cover the two squares in the top row at the extreme left. In the second case, it is impossible to cover the two squares in the bottom row at the extreme left. Therefore, it is impossible to tile the board with trominoes.

34. Such a board can be tiled with ij 2×3 rectangles of the form

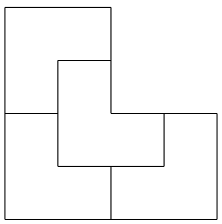
    

36. By symmetry, we may assume that the missing square is located in the 7×7 subboard shown in the following figure. Exercise 35 shows how to tile this subboard. Exercise 34 shows that the two 6×4 subboards can be tiled. Exercise 32 shows that the 5×5 subboard with a corner square can be tiled. Thus the deficient 11×11 board can be tiled with trominoes.



37. Basis Step ($n = 0$). In this case, the $2^n \times 2^n$ L-shape is a tromino and, so, it is tiled.

    Inductive Step. Assume that we can tile a $2^{n-1} \times 2^{n-1}$ L-shape with trominoes. Given a $2^n \times 2^n$ L-shape, divide it into four $2^{n-1} \times 2^{n-1}$ L-shapes:

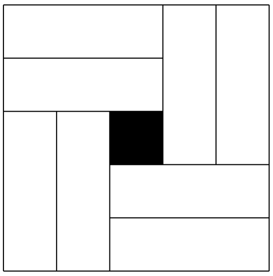


    By the inductive assumption, we can tile each of the four $2^{n-1} \times 2^{n-1}$ L-shapes with trominoes. The inductive step is complete.

40. Arguing as in the solution to Exercise 39, the numberings

1	2	3	1	2
3	1	2	3	1
2	3	1	2	3
1	2	3	1	2
3	1	2	3	1

show that the only possibility for the missing square is the center square. This board can be tiled:



41. An argument like those in the solutions to Exercises 39 and 40 shows that the only board that can be tiled with straight trominoes is the one with the missing square in row 3, column 3 (and the three boards symmetric to it).

43. We show only the inductive step. There are two cases: $a[k] < val$ and $a[k] \geq val$. If $a[k] \geq val$, the value of h does not change. Thus, we still have $a[p] < val$, for all p, $i < p \leq h$. After k is incremented, for all p, $h < p < k$, $a[p] \geq val$.

    If $a[k] < val$, then h is incremented and $a[h]$ and $a[k]$ are swapped. Let h_{old} denote the original value of h, and h_{new} denote the new (incremented) value of h. The value at h_{new} is the original $a[k]$. Since this value is less than val, the value of $a[h_{new}]$ is less than val. Thus, for all p, $i < p \leq h_{new}$, $a[p] < val$. After the swap, the value at k becomes h_{new}. By the inductive assumption, this value is greater than or equal to val. Thus after k is incremented, for all p, $h_{new} < p < k$, $a[p] \geq val$.

44. The argument is essentially identical to that of Example 2.4.7 that shows that any $2^n \times 2^n$ deficient board can be tiled with trominoes.

45. Notice that
$$k^3 - 1 = (k-1)[(k-2)(k-4) + 7(k-1)].$$
    Since 7 divides $k^3 - 1$, 7 divides $k - 1$ or $(k-2)(k-4) + 7(k-1)$. If 7 divides the latter expression, 7 also divides $(k-2)(k-4)$. If 7 divides $(k-2)(k-4)$, 7 divides either $k-2$ or $k-4$.

47. The Inductive Step fails if either a or b is 1. In this case, the inductive hypothesis is erroneously applied to the pair $a-1, b-1$, which includes a nonpositive integer.

48. To argue by contradiction, one must assume that the proposition fails *for some* $n \geq 2$. The alleged proof assumes that the proposition fails *for all* $n \geq 2$.

49. For $n = 2$, the inequality becomes $\frac{1}{2} + \frac{2}{3} < \frac{4}{3}$, which is true. Thus the Basis Step is true.

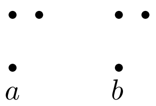
    Assume that the given statement holds for n. Now
$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} + \frac{n+1}{n+2} < \frac{n^2}{n+1} + \frac{n+1}{n+2}.$$
    The Inductive Step will be proved provided
$$\frac{n^2}{n+1} + \frac{n+1}{n+2} < \frac{(n+1)^2}{n+2}.$$

If we multiply the last inequality by $(n+1)(n+2)$, we obtain

$$n^2(n+2) + (n+1)^2 < (n+1)^3,$$

which is readily verified as true.

51. In the following figure



a and b are both survivors.

52. Suppose that there are three persons. The two persons closest together throw at each other, and the third person throws at one of the two closest. Therefore the third person survives. This complete the Basis Step.

Suppose that the assertion is true for n, and consider $n+2$ persons. Again, the closest pair throws at each other. There are now two cases to consider. If the remaining n persons all throw at one another, by the inductive assumption, there is a survivor. If at least one of the remaining n persons throws at one of the closest pair, among the remaining n persons, at most $n-1$ pies are thrown at one another. In this case, someone survives because there are not enough pies to go around. The Inductive Step is complete.

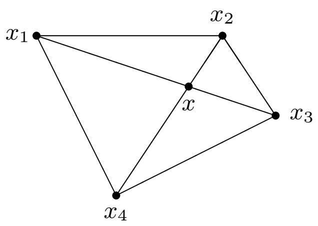
54. The statement is false. In the following figure

a throws a pie the greatest distance, but is not a survivor.

56. Let x_1 be a common point of X_2, X_3, X_4; let x_2 be a common point of X_1, X_3, X_4; let x_3 be a common point of X_1, X_2, X_4; and let x_4 be a common point of X_1, X_2, X_3. Since $x_1, x_2, x_3 \in X_4$, the triangle $x_1x_2x_3$ (perimeter and interior) is in X_4. Similarly, the triangle $x_1x_2x_4$ is in X_3; the triangle $x_1x_3x_4$ is in X_2; and the triangle $x_2x_3x_4$ is in X_1. We consider two cases:

Case 1: One of the points x_1, x_2, x_3, x_4 is in the triangle whose vertices are the other three points. For example, suppose that x_1 is in triangle $x_2x_3x_4$. Since triangle $x_2x_3x_4$ is in X_1, $x_1 \in X_1$. By definition, $x_1 \in X_2 \cap X_3 \cap X_4$. Therefore, $x_1 \in X_1 \cap X_2 \cap X_3 \cap X_4$.

Case 2: None of the points x_1, x_2, x_3, x_4 is in the triangle whose vertices are the other three points. In this case, x_1, x_2, x_3, x_4 are the vertices of a convex quadrilateral:



Now the intersection, x, of the diagonals of this quadrilateral belongs to each of the triangles and, thus, to each of X_1, X_2, X_3, X_4.

57. The proof is by induction on n. The Basis Step is $n = 4$, which is Exercise 56.

We turn to the Inductive Step. Assume that if $X_1, \ldots, X_n$ are convex sets, each three of which have a common point, then all n sets have a common point.

Let $X_1, \ldots, X_n, X_{n+1}$ be convex sets, each three of which have a common point. We must show that all $n + 1$ sets have a common point. By Exercise 55,

$$X_1, \ldots, X_{n-1}, X_n \cap X_{n+1} \tag{2.7}$$

are convex sets. We claim that any three of the sets in (2.7) have a common point. The claim is true by hypothesis if the three sets are any of $X_1, \ldots, X_{n-1}$. Consider $X_i, X_j, X_n \cap X_{n+1}$, $i < j \leq n-1$. By hypothesis, any three of X_i, X_j, X_n, X_{n+1} have a common point. By Exercise 56, X_i, X_j, X_n, X_{n+1} have a common point. Therefore, $X_i, X_j, X_n \cap X_{n+1}$ have a common point. Thus, any three of the sets in (2.7) have a common point. By the inductive assumption, the sets in (2.7) have a common point. The Inductive Step is complete.

59. We first prove the result for $n = 3$. Let A_1, A_2, A_3 be open intervals such that each pair has a nonempty intersection. Choose $x_1 \in A_1 \cap A_2$, $x_2 \in A_1 \cap A_3$, $x_3 \in A_2 \cap A_3$. Note that if any pair (x_1, x_2 or x_1, x_3 or x_3, x_3) is equal, it is in $A_1 \cap A_2 \cap A_3$. We may assume $x_1 < x_2$. We consider three cases. First suppose that $x_3 < x_1$. Since $x_2, x_3 \in A_3$, $[x_3, x_2] \subseteq A_3$. ($[a, b]$ is the set of all x satisfying $a \leq x \leq b$.) Thus $x_1 \in A_3$. Therefore $x_1 \in A_1 \cap A_2 \cap A_3$.

Next suppose that $x_1 < x_3 < x_2$. Since $x_1, x_2 \in A_1$, $[x_1, x_2] \subseteq A_1$. Thus $x_3 \in A_1$. Therefore $x_3 \in A_1 \cap A_2 \cap A_3$.

Finally suppose that $x_1 < x_2 < x_3$. Since $x_1, x_3 \in A_2$, $[x_1, x_3] \subseteq A_2$. Thus $x_2 \in A_2$. Therefore $x_2 \in A_1 \cap A_2 \cap A_3$. We have shown that if A_1, A_2, A_3 are open intervals such that each pair has a nonempty intersection, then $A_1 \cap A_2 \cap A_3$ is nonempty.

We now prove that given statement using induction on n. The Basis Step ($n = 2$) is trivial.

Assume that if $I_1, \ldots, I_n$ is a set of open intervals such that each pair has a nonempty intersection, then $I_1 \cap \cdots \cap I_n$ is nonempty. Let $I_1, \ldots, I_{n+1}$ be a set of open intervals such that each pair has a nonempty intersection. Since $I_n \cap I_{n+1}$ is nonempty, it is an open interval. We claim that

$$I_1, \ldots, I_{n-1}, I_n \cap I_{n+1}$$

is a set of open intervals such that each pair has a nonempty intersection. This is certainly true for pairs of the form I_i, I_j, $1 \leq i < j \leq n-1$. Consider a pair of the form I_i, $i \leq n-1$, and $I_n \cap I_{n+1}$. Since each pair among I_i, I_n, I_{n+1} has nonempty intersection, by the case $n = 3$ proved previously, $I_i \cap I_n \cap I_{n+1}$ is nonempty. Therefore,

$$I_1, \ldots, I_{n-1}, I_n \cap I_{n+1}$$

is a set of open intervals such that each pair has a nonempty intersection. By the inductive assumption

$$I_i \cap \cdots \cap I_{n-1} \cap (I_n \cap I_{n+1})$$

is nonempty. The inductive step is complete.

61. 5 62. 5

64. After j rounds, $2, 4, \ldots, 2j$ have been eliminated. At this point, there are 2^i persons. This is exactly the Josephus problem when the number of persons is a power of 2, except that the round begins with person $2j+1$, rather than with person 1. By Exercise 63, person $2j+1$ is the survivor.

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68. $\Delta a_n = a_{n+1} - a_n = (n+1)^2 - n^2 = 2n+1$. Let $b_n = \Delta a_n$. Then

$$\begin{aligned} b_1 + b_2 + \cdots + b_n &= (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + \cdots + (2n+1) \\ &= 2(1 + 2 + \cdots + n) + (1 + 1 + \cdots + 1) \\ &= 2(1 + 2 + \cdots + n) + n. \end{aligned}$$

By Exercise 67,

$$b_1 + b_2 + \cdots + b_n = a_{n+1} - a_1 = (n+1)^2 - 1^2 = n^2 + 2n.$$

Combining the previous equations, we obtain

$$n^2 + 2n = 2(1 + 2 + \cdots + n) + n.$$

Solving for $1 + 2 + \cdots + n$, we obtain

$$1 + 2 + \cdots + n = \frac{n^2 + 2n - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

69. Let $a_n = n!$. Then

$$\Delta a_n = a_{n+1} - a_n = (n+1)! - n! = n![(n+1) - 1] = n(n!).$$

Let $b_n = \Delta a_n$. Then

$$b_1 + b_2 + \cdots + b_n = 1(1!) + 2(2!) + \cdots + n(n!).$$

By Exercise 67,

$$b_1 + b_2 + \cdots + b_n = a_{n+1} - a_1 = (n+1)! - 1!.$$

Combining the previous equations, we obtain

$$1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1.$$

71. Since p is divisible by k, there exists t_1 such that $p = t_1 k$. Since q is divisible by k, there exists t_2 such that $p = t_2 k$. Now

$$p + q = t_1 k + t_2 k = (t_1 + t_2)k.$$

Therefore, $p + q$ is divisible by k.

Problem-Solving Corner: Mathematical Induction

1. The Basis Step ($n = 0$) is $H_1 \leq 1 + 0$. Since $H_1 = 1$, the Basis Step is true.
Now assume that $H_{2^n} \leq 1 + n$. Then

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &\leq 1 + n + \frac{1}{2^n+1} + \cdots + \frac{1}{2^n+1} \\ &= 1 + n + \frac{2^n}{2^n+1} \leq 1 + (n+1). \end{aligned}$$

The Inductive Step is complete.

2. The Basis Step ($n = 1$) is $H_1 = 2H_1 - 1$. Since $H_1 = 1$, the Basis Step is true.
Now assume that

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

Then

$$\begin{aligned} H_1 + H_2 + \cdots + H_n + H_{n+1} &= (n+1)H_n - n + H_{n+1} \\ &= (n+1)\left(H_{n+1} - \frac{1}{n+1}\right) \qquad \text{by Exercise 3} \\ &\quad - n + H_{n+1} \\ &= (n+2)H_{n+1} - (n+1). \end{aligned}$$

The Inductive Step is complete.

3. $H_{n+1} - \dfrac{1}{n+1} = \left(\dfrac{1}{1} + \dfrac{1}{2} + \cdots + \dfrac{1}{n} + \dfrac{1}{n+1}\right) - \dfrac{1}{n+1} = \dfrac{1}{1} + \dfrac{1}{2} + \cdots + \dfrac{1}{n} = H_n$

4. We prove the assertion by induction. The Basis Step is $n = 1$:

$$1 \cdot H_1 = 1 = \frac{3}{2} - \frac{1}{2} = \frac{1 \cdot 2}{2} H_2 - \frac{1 \cdot 2}{4}.$$

For the Inductive Step, assume the assertion if true for n. Now

$$\begin{aligned} 1 \cdot H_1 + \cdots + nH_n + (n+1)H_{n+1} &= \frac{n(n+1)}{2} H_{n+1} - \frac{n(n+1)}{4} + (n+1)H_{n+1} \\ &= (n+1)H_{n+1}\left[\frac{n}{2} + 1\right] - \frac{n(n+1)}{4} \\ &= H_{n+1}\left[\frac{(n+1)(n+2)}{2}\right] - \frac{n(n+1)}{4} \\ &= \left[H_{n+2} - \frac{1}{n+2}\right]\left[\frac{(n+1)(n+2)}{2}\right] \qquad \text{by Exercise 3} \\ &\quad - \frac{n(n+1)}{4} \\ &= H_{n+2}\left[\frac{(n+1)(n+2)}{2}\right] - \frac{n+1}{2} - \frac{n(n+1)}{4} \\ &= H_{n+2}\left[\frac{(n+1)(n+2)}{2}\right] - \frac{(n+1)(n+2)}{4}. \end{aligned}$$

Section 2.5

2. Verify directly the cases $n = 24, \ldots, 28$. Assume that the statement is true for postage i satisfying $24 \leq i < n$. We must show that we can make n cents postage using only 5-cent and 7-cent stamps. We may assume that $n > 28$. Then $n > n - 5 > 23$. By the inductive assumption, we can make $n - 5$ cents postage using 5-cent and 7-cent stamps. Add a 5-cent stamp to obtain n cents postage.

4. The Basis Step ($n = 6$) is proved by using three 2-cent stamps. Now assume that we can make postage for n cents. If there is at least one 7-cent stamp, replace it by four 2-cent stamps to make $n+1$ cents postage. If there are no 7-cent stamps, there are at least three 2-cent stamps (because $n \geq 6$). Replace three 2-cent stamps by one 7-cent stamp to make $n+1$ cents postage. The Inductive Step is complete.

5. The Basis Step ($n = 24$) is proved by using two 5-cent stamps and two 7-cent stamps. Now assume that we can make postage for n cents. If there are at least two 7-cent stamps, replace two 7-cent stamps with three 5-cent stamps to make $n+1$ cents postage. If there is exactly one 7-cent stamp, then there are at least four 5-cent stamps (because $n \geq 24$). Replace one 7-cent stamp and four 5-cent stamps with four 7-cent stamps to make $n+1$ cents postage. If there are no 7-cent stamps, then there are at least five 5-cent stamps (again because $n \geq 24$). Replace five 5-cent stamps with three 7-cent stamps and one 5-cent to make $n + 1$ cents postage. The Inductive Step is complete.

7. We must have $4 \leq \lfloor n/2 \rfloor$. Since this inequality fails for $n = 5, 6, 7$, the Basis Steps are $n = 4, 5, 6, 7$.

8. We must have $2 \leq \lfloor n/3 \rfloor$. Since this inequality fails for $n = 3, 4, 5$, the Basis Steps are $n = 2, 3, 4, 5$.

10. We omit the Basis Step. For the Inductive Step, we have

$$c_n = c_{\lfloor n/2 \rfloor} + n^2 < 4\left\lfloor \frac{n}{2} \right\rfloor^2 + n^2 \leq 4\left(\frac{n}{2}\right)^2 + n^2 = 2n^2 < 4n^2.$$

12. We omit the Basis Step. For the Inductive Step, we have

$$\begin{aligned} c_n = 4c_{\lfloor n/2 \rfloor} + n &\leq 4[4(\lfloor n/2 \rfloor - 1)^2] + n \\ &\leq 4[4(n/2 - 1)^2] + n \\ &= 4n^2 - 15n + 16 \\ &\leq 4(n-1)^2. \end{aligned}$$

The last inequality reduces to $12 \leq 7n$, which is true since $n > 1$.

13. We omit the Basis Steps ($n = 2, 3$). We turn to the Inductive Step. Assume that $n \geq 4$. Then $n/2 \geq 2$, so $\lfloor n/2 \rfloor \geq 2$. Then

$$\begin{aligned} c_n = 4c_{\lfloor n/2 \rfloor} + n &> 4(\lfloor n/2 \rfloor + 1)^2/8 + n \\ &\geq 4[(n-1)/2 + 1]^2/8 + n \\ &= (n+1)^2/8 + n \\ &> (n+1)^2/8. \end{aligned}$$

We used the fact that $\lfloor n/2 \rfloor \ge (n-1)/2$ for all n.

15. BASIS STEP ($n = 1$). The first player removes one card (from either pile). The second player then removes the last card and wins the game.

    INDUCTIVE STEP. Suppose that $n > 1$ and whenever there are two piles of $k < n$ cards, the second player can always win the game.

    Suppose that there are two piles of n cards. The first player removes i cards from one of the piles. If $i = n$ (i.e., the first player removes all of the cards from one pile), the second player can win by removing all of the cards from the remaining pile. If $i < n$, the second player removes i cards from the other pile leaving two piles each with $n - i$ cards. The game then resumes with the first player facing two piles each with $k = n - i < n$ cards. By the inductive assumption, the second player can win the game. The inductive proof is complete.

17. $q = -6$, $r = 7$ 18. $q = 0$, $r = 7$ 20. $q = 0$, $r = 0$ 21. $q = 1$, $r = 0$

23. If
$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$
where $n_1 < n_2 < \ldots < n_k$, another representation is
$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_{k-1}} + \frac{1}{n_k + 1} + \frac{1}{n_k(n_k + 1)}$$

24. (b) Since $p/q < 1$, $n > 1$. Since n is the smallest positive integer satisfying $1/n \le p/q$ and $n - 1$ is a positive integer less than n, $p/q < 1/(n-1)$.

    (d) We have
$$\frac{p_1}{q_1} = \frac{np - q}{nq} = \frac{p}{q} - \frac{1}{n}. \tag{2.8}$$
Since $1/n < p/q$, equation (2.8) shows that
$$0 < \frac{p_1}{q_1}.$$
Since
$$\frac{p}{q} < \frac{1}{n-1},$$
we have
$$np - p < q$$
or
$$p_1 = np - q < p.$$
The third inequality is established.
Now
$$\frac{p_1}{q_1} < \frac{p}{q_1} = \frac{p}{nq} = \frac{1}{n}\frac{p}{q} < \frac{1}{n} \cdot 1 = \frac{1}{n}. \tag{2.9}$$

In particular,

$$\frac{p_1}{q_1} < 1.$$

We have established the second inequality.

By the inductive assumption, p_1/q_1 can be expressed in Egyptian form. The last equation follows.

(e) See (2.9).

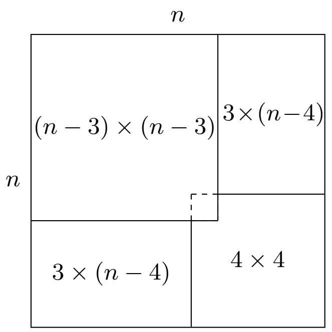
(f) The equation is true because of (d). For any $i = 1, \ldots, k$,

$$\frac{1}{n_i} \leq \frac{1}{n_1} + \cdots + \frac{1}{n_k} = \frac{p_1}{q_1} < \frac{1}{n}.$$

It follows that $n, n_1, \ldots, n_k$ are distinct.

25. $\frac{3}{8} = \frac{1}{3} + \frac{1}{24}$, $\frac{5}{7} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70}$, $\frac{13}{19} = \frac{1}{2} + \frac{1}{6} + \frac{1}{57}$

28. Enclose the missing square in a corner $(n-3) \times (n-3)$ subboard as shown in the following figure. Since 3 divides $n^2 - 1$, 3 also divides $(n-3)^2 - 1$. Now $n - 3$ is odd, $n - 3 > 5$, and 3 divides $(n-3)^2 - 1$, so by Exercise 27, we may tile this subboard. Tile the two $3 \times (n-4)$ subboards using the result of Exercise 34, Section 2.4. Tile the deficient 4×4 subboard using Example 2.4.7. The $n \times n$ board is tiled.



29. If $n = 0$, $d \cdot 1 = d > 0$, and 1 is in X. If $n > 0$, $d(2n) = n(2d) > n$; thus $2n$ is in X. In either case X is nonempty. Since $d > 0$ and $n \geq 0$, X contains only positive integers. By the Well-Ordering Property, X contains a least element $q' > 0$. Then $dq' > n$. Let $q = q' - 1$. We cannot have $dq > n$ (for then q' would not be the *least* element in X); therefore, $dq \leq n$. Let $r = n - dq$. Then $r \geq 0$. Also

$$r = n - dq = n - d(q' - 1) < dq' - d(q' - 1) = d.$$

Therefore, we have found q and r satisfying

$$n = dq + r \qquad 0 \leq r < d.$$

30. We first prove Theorem 2.5.6 for $n > 0$. The Basis Step is $n = 1$. If $d = 1$, we have $n = dq + r$, where $q = n$ and $r = 0$, $0 \leq r < d$. If $d > 1$, we have $n = dq + r$, where $q = 0$ and $r = 1$, $0 \leq r < d$. Thus Theorem 2.5.6 is true for $n = 1$.

Assume that Theorem 2.5.6 holds for n. Then there exists q' and r' such that

$$n = dq' + r' \qquad 0 \leq r' < d.$$

Now

$$n + 1 = dq' + (r' + 1).$$

If $r' < d - 1$, then $r' + 1 < d$. In this case, if we take $q = q'$ and $r = r' + 1$, we have

$$n + 1 = dq + r \qquad 0 \leq r < d.$$

If $r' = d - 1$, we have

$$n + 1 = d(q' + 1).$$

In this case, if we take $q = q' + 1$ and $r = 0$, we have

$$n + 1 = dq + r \qquad 0 \leq r < d.$$

The Inductive Step is complete. Therefore, Theorem 2.5.6 is true for all $n > 0$.

If $n = 0$, we may write

$$n = dq + r,$$

where $q = r = 0$. Therefore, Theorem 2.5.6 is true for $n = 0$.

Finally, suppose that $n < 0$. Then $-n > 0$, so by the first part of the proof, there exist q' and r' such that

$$-n = dq' + r' \qquad 0 \leq r' < d.$$

If $r' = 0$, we may take $q = -q'$ and $r = 0$ to obtain

$$n = dq + 0.$$

If $r' > 0$, we take $q = -q' - 1$ and $r = d - r'$. Then $0 < r < d$ and

$$n = d(-q') - r' = d(q + 1) + (r - d) = dq + r.$$

Therefore, Theorem 2.5.6 is true for $n < 0$.

32. Suppose that we have a propositional function $S(n)$ whose domain of discourse is the set of integers greater than or equal to n_0. Suppose that $S(n_0)$ is true and, for all $n > n_0$, if $S(k)$ is true for all k, $n_0 \leq k < n$, then $S(n)$ is true. We must prove that $S(n)$ is true for every integer $n \geq n_0$. We first assume that $n_0 \geq 0$.

    We argue by contradiction. So assume that $S(n)$ is false for some integer $n_1 \geq n_0$. Let X be the set of nonnegative integers for which $S(n)$ is false. Then X is nonempty. By the Well-Ordering Property, X has a least element n_2. Since $S(n_0)$ is true, $n_2 > n_0$. Furthermore, for any k, $n_0 \leq k < n_2$, $S(k)$ is true [otherwise n_2 would not be the least integer n for which $S(n)$ is false]. Since $S(k)$ is true for all k, $n_0 \leq k < n_2$, by hypothesis, $S(n_2)$ is true. Contradiction.

    If $n_0 < 0$, apply the previous argument to the propositional function

$$S'(n)\colon\ S(n + n_0)$$

    with domain of discourse the set of nonnegative integers.

33. The strong form of induction clearly implies the form of induction where the Inductive Step is: "If $S(n)$ is true, then $S(n + 1)$ is true." For the converse, use Exercises 31 and 32.