CHAPTER 2

First-Order Differential Equations

2.1 SEPARABLE EQUATIONS

According to historians of mathematics, Leibniz *implicitly* discovered the method of separation of variables in 1691, but John Bernoulli should be credited with the *explicit* process and the name (*separatio indeterminatarum*) in 1694. [See *Ordinary Differential Equations* by E. L. Ince (Dover Publications, 1956).]

Determining whether a differential equation is separable is often a test of a student's algebraic skills. A paper by D. Scott [*Amer. Math. Monthly* **92** (1985), 422–423] contains conditions that ensure separability: Suppose that in D, an open convex set in the plane [an open disk in the x-y plane will do], f, f_x, f_y, and f_{xy} exist and are continuous, $f(x,y) \neq 0$, and $f(x,y)f_{xy}(x,y) = f_x(x,y)f_y(x,y)$ holds, then there exist continuously differentiable functions $g(x)$ and $h(y)$ such that, for every (x,y) in $D, f(x,y) = g(x)h(y)$. Here the subscripts denote partial derivatives. A partial converse holds. Try the theorem on Exercise A10 or on $f(x,y) = 2x^2 + y - x^2y + xy - 2x - 2$, which is separable.

The method of separation of variables is the one most students may have seen in a calculus course. I find it useful to review partial fractions via examples such as Example 2.1.6. You may want to introduce students to the partial fractions capabilities of your CAS. *Maple*, for example, has the **convert** command, with the **parfrac** option. Also, I emphasize the need to watch out for *singular* solutions. With respect to the *autonomous equations* introduced in Section 2.4, these singular solutions are *equilibrium solutions* (Section 2.6), important for the qualitative analysis of differential equations. I have not seen a CAS ODE solver that produces singular solutions.

The substitution methods described after Exercises A11 and A14 are two classical change of variable techniques (tricks?) for converting certain equations whose variables are not separable into separable equations.

Exercise C3 concerns the *catenary*. Exercise C6 asks the student to work analytically with a basic form of the *logistic equation*, which will be discussed qualitatively in Section 2.5.

A

1. $\frac{dy}{dx} = \frac{A-2y}{x} \Rightarrow \frac{dy}{A-2y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{A-2y} = \int \frac{dx}{x} \Rightarrow -\frac{1}{2}\ln|A-2y| = \ln|x| + C_1 \Rightarrow \ln|A-2y| = -2\ln|x| + C_2 = \ln\left(\frac{1}{x^2}\right) + C_2 \Rightarrow$ (exponentiating)$|A-2y| = \frac{C_3}{x^2}$, where $C_3 > 0 \Rightarrow A - 2y = \frac{C_4}{x^2}$, where C_4 is arbitrary $\Rightarrow y = \frac{1}{2}\left(A - \frac{C_4}{x^2}\right) = \frac{A}{2} + \frac{C}{x^2}$. Note that $C = -C_4/2$ can be *any* real number if C_4 can be any real number.

2. $\frac{dy}{dx} = \frac{-xy}{x+1} \Rightarrow \frac{dy}{y} = \frac{-x\,dx}{x+1} \Rightarrow \int \frac{dy}{y} = -\int \frac{x\,dx}{x+1} = -\int \frac{(x+1)-1}{x+1}dx = -\int\left(1 - \frac{1}{x+1}\right)dx = -\int 1dx + \int \frac{dx}{x+1} \Rightarrow \ln|y| = -x + \ln|x+1| + C_1 \Rightarrow |y| = e^{-x}\cdot e^{\ln|x+1|}\cdot e^{C_1} = C_2|x+1|e^{-x} \Rightarrow y = C(x+1)e^{-x}$.

3. $y' = 3\sqrt[3]{y^2} \Rightarrow \frac{dy}{dt} = 3\left(y^{\frac{2}{3}}\right) \Rightarrow \frac{dy}{y^{\frac{2}{3}}} = 3\,dt \Rightarrow \int y^{-\frac{2}{3}}dy = 3\int 1\,dt \Rightarrow 3y^{\frac{1}{3}} = 3t + C_1 \Rightarrow y^{\frac{1}{3}} = t + C_2 \Rightarrow y = (t+C_2)^3$. Now the initial condition $y(2) = 0$ implies that $0 = (0+C_2)^3$, so that $C_2 = -2$. Therefore, $y = (t-2)^3 = t^3 - 6t^2 + 12t - 8$. But notice that in separating the variables we divided by a power of y. The solution $y \equiv 0$ is a **singular solution** of the basic ODE and satisfies the initial condition.

4. $\frac{dy}{dx} = \frac{(y-1)(y-2)}{x} \Rightarrow \frac{dy}{(y-1)(y-2)} = \frac{dx}{x} \Rightarrow \int \frac{dy}{(y-1)(y-2)} = \int \frac{dx}{x} \Rightarrow \int\left(\frac{1}{y-2} - \frac{1}{y-1}\right)dy = \int \frac{dx}{x} \Rightarrow \ln|y-2| - \ln|y-1| = \ln|x| + C_1 \Rightarrow \ln\left|\frac{y-2}{y-1}\right| = \ln|x| + C_1 \Rightarrow \left|\frac{y-2}{y-1}\right| = C_2|x| \Rightarrow \frac{y-2}{y-1} = Cx \Rightarrow y = \frac{2-Cx}{1-Cx}$. Since we divided by $y-1$ and $y-2$, we must check for singular solutions. The solution $y \equiv 2$ can be obtained by choosing $C = 0$, but there is no value of C for which $(2-Cx)/(1-Cx) = 1$. Thus $y \equiv 1$ is a **singular solution**.

5. $(\cot x)y' + y = 2 \Rightarrow (\cot x)\frac{dy}{dx} = 2 - y \Rightarrow \frac{dy}{2-y} = \frac{dx}{\cot x} = \tan x\,dx \Rightarrow \int \frac{dy}{2-y} = \int \tan x\,dx \Rightarrow -\ln|2-y| = -\ln|\cos x| + C_1 \Rightarrow \ln|2-y| = \ln|\cos x| + C_2 \Rightarrow |2-y| = C_3|\cos x| \Rightarrow 2 - y = C_4\cos x$, so that $y = 2 - C\cos x$. The initial condition implies that $-1 = y(0) = 2 - C\cos(0) = 2 - C$, so that $C = 3$ and $y = 2 - 3\cos x$. The only possible singular solution is $y \equiv 2$, but this can be obtained by letting $C = 0$.

6. $\frac{dx}{dt} = -\frac{\sin t\cos^2 x}{\cos^2 t} \Rightarrow \frac{dx}{\cos^2 x} = -\frac{\sin t}{\cos^2 t}dt \Rightarrow \int \frac{dx}{\cos^2 x} = \int -\frac{\sin t}{\cos^2 t}dt \Rightarrow \int \sec^2 x\,dx = \int(\cos t)^{-2}(-\sin t)dt \Rightarrow \tan x = -(\cos t)^{-1} + C_1$. At this point we can use the initial condition: $\tan x(0) = -(\cos 0)^{-1} + C_1$, or $\tan 0 = 0 = -1 + C_1$, so that $C_1 = 1$. Then $\tan x = -(\cos t)^{-1} + 1 \Rightarrow x(t) = \arctan\left(-\frac{1}{\cos t} + 1\right) = \arctan\left(\frac{-1+\cos t}{\cos t}\right) = -\arctan\left(\frac{1-\cos t}{\cos t}\right)$.

7. $x^2y^2y' + 1 = y \Rightarrow x^2y^2\frac{dy}{dx} = y - 1 \Rightarrow \frac{y^2}{y-1}dy = \frac{dx}{x^2} \Rightarrow \int \frac{y^2}{y-1}dy = \int \frac{dx}{x^2} \Rightarrow \int\left(y + 1 + \frac{1}{y-1}\right)dy = \int x^{-2}dx \Rightarrow \frac{y^2}{2} + y + \ln|y-1| = -\frac{1}{x} + C$. The constant function $y \equiv 1$ is a **singular solution**. (We divided by $y - 1$ earlier. Notice that the implicit solution formula is not defined for $y = 1$.)

8. $xy' + y = y^2 \Rightarrow x\frac{dy}{dx} = y^2 - y = y(y-1) \Rightarrow \frac{dy}{y(y-1)} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y(y-1)} = \int \frac{dx}{x} \Rightarrow \int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \int \frac{dx}{x} \Rightarrow \ln|y-1| - \ln|y| = \ln|x| + C_1 \Rightarrow \ln\left|\frac{y-1}{y}\right| = \ln|x| + C_1 \Rightarrow \left|\frac{y-1}{y}\right| = C_2|x|$, where $C_2 > 0 \Rightarrow \frac{y-1}{y} = Cx \Rightarrow y = \frac{1}{1-Cx}$. Using the initial condition, we find that $0.5 = y(1) = 1/(1-C)$, so that $C = -1$ and $y = \frac{1}{1+x}$. Note that the basic equation has $y \equiv 0$ as a **singular solution**. In separating variables, we divided by $y(y-1)$, but $y = 1$ can be obtained by letting $C = 0$ in the one-parameter solution formula. These constant solutions don't satisfy the given initial condition, however.

9. $\frac{dz}{dx} = 10^{x+z} = 10^x 10^z \Rightarrow \frac{dz}{10^z} = 10^x dx \Rightarrow \int \frac{dz}{10^z} = \int 10^x dx \Rightarrow -\frac{1}{\ln 10}10^{-z} = \frac{1}{\ln 10}10^x + C_1 \Rightarrow 10^{-z} = -10^x + C \Rightarrow -z\ln 10 = \ln\left(C - 10^x\right) \Rightarrow z = \frac{\ln(C-10^x)}{\ln 10}$. Note that for each particular value of the parameter C, the solution is defined only for $10^x < C$—that is, for $x < \ln C/\ln 10$ (or $x < \log_{10} C$).

10. $\frac{dy}{dx} = 1 + x + y^2 + xy^2 = 1 + x + y^2(1+x) = (1+x)\left(1+y^2\right) \Rightarrow \frac{dy}{1+y^2} = (1+x)dx \Rightarrow \int \frac{dy}{1+y^2} = \int (1+x)dx \Rightarrow \arctan y = x + \frac{x^2}{2} + C \Rightarrow y = \tan\left(x + \frac{x^2}{2} + C\right)$.

11. $(y')^2 + (x+y)y' + xy = 0 \Rightarrow (y'+x)(y'+y) = 0 \Rightarrow y' + x = 0$ or $y' + y = 0 \Rightarrow \frac{dy}{dx} = -x$ or $\frac{dy}{dx} = -y \Rightarrow y = -\frac{x^2}{2} + C$ or $y = Ce^{-x}$.

12. $y' - y = 2x - 3 \Rightarrow y' = y + 2x - 3$. Letting $z = y + 2x - 3$, we have $\frac{dz}{dx} = y' + 2 = (y + 2x - 3) + 2 = z + 2$. Separating variables, we see that $\frac{dz}{z+2} = dx$, and integrating gives us $\ln|z+2| = x + C_1$, $|z+2| = C_2e^x$, $z + 2 = Ce^x$, so that $z = Ce^x - 2$. Replacing z by $y + 2x - 3$, we conclude that $y = Ce^x - 2x + 1$.

13. $(x+2y)y' = 1 \Rightarrow y' = \frac{1}{x+2y}$. Letting $z = x+2y$, we have $\frac{dz}{dx} = 1+2y' = 1+\frac{2}{x+2y} = 1+\frac{2}{z} = \frac{z+2}{z}$. Separating variables, we get $\frac{z}{z+2}dz = dx$, or $\left(1 - \frac{2}{z+2}\right)dz = dx$, and integrating gives us $z - 2\ln|z+2| = x + C_1$. Replacing z by $x + 2y$, we have $x + 2y - 2\ln|x + 2y + 2| = x + C_1$.

There is no member of this one-parameter family of solutions that satisfies the Initial condition $y(0) = -1$. However, because we divided by $z + 2 = x + 2y + 2$ in separating variables above, we have a singular solution $y = -(x+2)/2$, which also satisfies the initial condition.

14. $y' = \sqrt{4x + 2y - 1}$. Letting $z = 4x + 2y - 1$, we have $\frac{dz}{dx} = 4 + 2y' = 4 + 2\sqrt{4x + 2y - 1} = 4 + 2\sqrt{z}$. Separating variables and integrating, we have $\int \frac{dz}{4+2\sqrt{z}} = \int dx$. If we let $z = u^2$ in the first integral, then $dz = 2u\,du$ and the last equation becomes $\int \frac{2u}{4+2u}du = \int dx$, $\int \frac{u}{2+u}du = \int dx$, $\int\left(1 - \frac{2}{2+u}\right)du = x + C_1$, $u - 2\ln|2+u| = x + C_1$. Replacing u by $\sqrt{z} = \sqrt{4x + 2y - 1}$, we get our final answer: $\sqrt{4x + 2y - 1} - 2\ln\left|\sqrt{4x + 2y - 1} + 2\right| = x + C$. Of course, you could have let $z^2 = 4x + 2y - 1$ immediately, so that $2z\frac{dz}{dx} = 4 + 2y' = 4 + 2z$, etc.

15. $y' = \frac{x+y}{x-y} = \frac{x\left(1+\frac{y}{x}\right)}{x\left(1-\frac{y}{x}\right)} = \frac{\left(1+\frac{y}{x}\right)}{\left(1-\frac{y}{x}\right)}$. Now let $z = y/x$. As in the example, $y' = z + x\left(\frac{dz}{dx}\right)$, so that the original equation becomes $z + x\left(\frac{dz}{dx}\right) = \frac{1+z}{1-z}$, or $x\left(\frac{dz}{dx}\right) = \frac{1+z}{1-z} - z = \frac{1+z^2}{1-z}$. Separating variables, we get $\left(\frac{1-z}{1+z^2}\right)dz = \left(\frac{1}{1+z^2} - \frac{z}{1+z^2}\right)dz = \frac{dx}{x}$. Integrating, we find that $\arctan z - \frac{1}{2}\ln\left|1+z^2\right| = \ln|x| + C$. Replacing z by y/x, we get the solution $\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) - \ln|x| - C = 0$.

16. $\dot{x} = \frac{t-3x}{3t+x} = \frac{t(1-3\frac{x}{t})}{t(3+\frac{x}{t})} = \frac{(1-3\frac{x}{t})}{(3+\frac{x}{t})}$. Now let $z = x/t$, so that $\frac{dx}{dt} = z + t\left(\frac{dz}{dt}\right)$ and the equation becomes $z + t\left(\frac{dz}{dt}\right) = \frac{1-3z}{3+z}$, or $t\left(\frac{dz}{dt}\right) = \frac{1-6z-z^2}{3+z}$. Separating variables and integrating, we get $\int \frac{z+3}{z^2+6z-1}dz = -\int \frac{dt}{t}$, or $\frac{1}{2}\ln\left|z^2+6z-1\right| = -\ln|t| + C_1, \ln\left|z^2+6z-1\right| = -2\ln|t| + C_2, \left|z^2+6z-1\right| = \frac{C_3}{t^2}, z^2+6z-1 = \frac{C_4}{t^2}$. Replacing z by x/t and simplifying, we get $x^2 + 6xt - t^2 = C$. This equation describes a family of hyperbolas. Alternatively, looking at this as a quadratic equation in x we can use the quadratic formula to find that $x = \frac{-6t \pm \sqrt{(6t)^2 - 4(1)(-t^2-C)}}{2} = \frac{-6t \pm \sqrt{40t^2+4C}}{2} = \frac{-6t \pm 2\sqrt{10t^2+C}}{2} = -3t \pm \sqrt{10t^2+C}$. Therefore, we have two one-parameter families of solutions: $x(t) = -3t + \sqrt{10t^2+C}$ and $x(t) = -3t - \sqrt{10t^2+C}$.

17. $y' = \frac{x}{y} + \frac{y}{x}$. We have a choice here: Let $z = x/y$ or $z = y/x$. Because it will make our work a little easier, we choose $z = y/x$. Now $\frac{dy}{dx} = z + x\left(\frac{dz}{dx}\right)$ allows us to write our original equation as $\frac{dy}{dx} = z + x\left(\frac{dz}{dx}\right) = \frac{1}{z} + z$, or $x\left(\frac{dz}{dx}\right) = \frac{1}{z}$. Separating variables and integrating, we find that $\frac{z^2}{2} = \ln|x| + C_1$. Substituting y/x for z and multiplying by 2, we get $\left(\frac{y}{x}\right)^2 = 2\ln|x| + C_2, y^2 = 2x^2\ln|x| + C_2x^2$, so that we have two one-parameter families of solutions: $y = x\sqrt{2\ln|x| + C}$ and $y = -x\sqrt{2\ln|x| + C}$.

18. $\frac{dy}{dx} = \frac{y^2+2xy-x^2}{x^2+2xy-y^2} = \frac{x^2\left(\frac{y^2}{x^2}+\frac{2y}{x}-1\right)}{x^2\left(1+\frac{2y}{x}-\frac{y^2}{x^2}\right)} = \frac{\left(\frac{y^2}{x^2}+\frac{2y}{x}-1\right)}{\left(1+\frac{2y}{x}-\frac{y^2}{x^2}\right)}$. If we let $z = y/x$, the equation becomes $z + x\left(\frac{dz}{dx}\right) = \frac{z^2+2z-1}{1+2z-z^2}$, or $x\left(\frac{dz}{dx}\right) = \frac{z^3-z^2+z-1}{1+2z-z^2}$. Separating variables and integrating, we get $\int \frac{1+2z-z^2}{z^3-z^2+z-1}dz = \int \frac{1}{x}dx, \int\left(\frac{1}{z-1} - \frac{2z}{z^2+1}\right)dz = \int \frac{1}{x}dx, \ln|z-1| - \ln\left|z^2+1\right| = \ln|x| + C_1$, and $\ln\left|\frac{z-1}{z^2+1}\right| = \ln|x| + C_1$, so that $\frac{z-1}{z^2+1} = Cx$. Now replacing z by y/x, we get the result that $C(x^2+y^2) + x - y = 0$. By using some algebra (completing squares, etc.), you can see that this equation describes a family of circles passing through the origin whose centers are on the line $y = x$. For each value of C, the radius of the circle is $\sqrt{2}/2|C|$. Alternatively, you can view the equation $C(x^2+y^2) + x - y = 0$ as a quadratic equation in y and solve it, obtaining two one-parameter families of solutions.

B

1. The FTC says that $\frac{d}{dx}f(x) = \frac{d}{dx}\int_0^x f(t)dt = f(x)$. Since $f(0) = \int_0^0 f(t)dt = 0$, we see that we have the IVP $y' = y, y(0) = 0$. Solving the equation by separating variables, we find that $y = Ce^x$. The initial condition implies that $C = 0$, so that $y = f(x) \equiv 0$.

2. a. $\frac{dx}{dt} = \frac{x^2+x}{t} \Rightarrow \frac{dx}{x(x+1)} = \frac{dt}{t} \Rightarrow \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \int \frac{dt}{t}$, so that we have $\ln|x| - \ln|x+1| = \ln|t| + C_1$, or $\ln\left|\frac{x}{x+1}\right| = \ln|t| + C_1$. Thus $\left|\frac{x}{x+1}\right| = C_2|t|$, or $\frac{x}{x+1} = C_3t$. Solving this last equation for x, we get $x = \frac{Ct}{1-Ct}$.

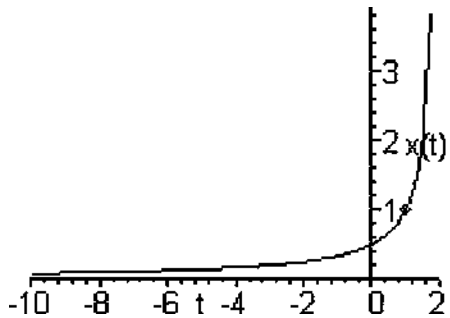
b. First of all, we can't find such a particular solution using the formula for the one-parameter family of solutions found in (a): $-1 = x(0) = C(0)/(1 - C(0))$ implies that $-1 = 0$. Basically, the right-hand side of the original differential equation is not defined anywhere in the t-x plane where $t = 0$. However, the answer to (c) below shows us how to deal with this problem.

c. In separating variables in (a), we divided by $x+1$, which vanishes when $x = -1$. This means that we implicitly assumed that x was not equal to -1. Since $x \equiv -1$ is easily seen to be a solution, we have $x \equiv -1$ as a **singular solution**—one that happens to satisfy the initial condition $x(0) = -1$ specified in part (b).

3. a. $\frac{dx}{dt} = x^2 \Rightarrow x^{-2}dx = dt \Rightarrow -x^{-1} = t + C_1 \Rightarrow \frac{1}{x} = -t + C_2 \Rightarrow x(t) = \frac{1}{C-t}$. Now $x(1) = 1 \Rightarrow 1 = \frac{1}{C-1} \Rightarrow C = 2 \Rightarrow x(t) = \frac{1}{2-t}$.

b. The interval I can be as large as $(-\infty, 2)$ or $(2, \infty)$. Any such interval I cannot include the point $t = 2$, at which $x(t)$ is not defined.

c.



d. Using the one-parameter formula found in (a), we want $0 = x(0) = \frac{1}{C-0} = \frac{1}{C}$, which is impossible. However, we notice that $x \equiv 0$ is a **singular solution** that satisfies the initial condition $x(0) = 0$.

4. a. $\frac{dQ}{dP} = -\frac{cQ}{1+cP} \Rightarrow \frac{dQ}{Q} = -\frac{c}{1+cP}dP$, $\int \frac{dQ}{Q} = -\int \frac{c}{1+cP}dP$, $\ln|Q| = -\ln|1+cP| + C_1 \Rightarrow |Q| = \frac{e^{C_1}}{|1+cP|} \Rightarrow Q = \frac{K}{1+cP}$, where $K = \pm e^{C_1}$.

b. If $Q = \frac{K}{1+cP}$, then $Q(0) = K$ and $Q(1) = \frac{K}{1+c}$, so $2 \approx \frac{Q(0)}{Q(1)} = \frac{K}{\frac{K}{1+c}} = 1 + c$ implies that $c \approx 1$.

c. Using the approximate value of c from part (b), we find that $Q(0.20) = \frac{K}{1+(1)(0.20)} = \frac{K}{1.2} = \frac{5}{6}K = \frac{5}{6}Q(0)$. This result tells us that the cost of national health insurance when individuals pay 20% of their health services cost is five-sixths the cost when individuals pay nothing "out of pocket."

5. $\frac{dy}{dt} = -\frac{\ln 2}{30}(y - 20) \Rightarrow \frac{dy}{y-20} = -\frac{\ln 2}{30}dt \Rightarrow \ln|y - 20| = -\frac{\ln 2}{30}t + C_1 \Rightarrow |y - 20| = C_2 e^{-\frac{\ln 2}{30}t} = C_2\left(e^{\ln 2^{-1/30}}\right)^t = C_2\left(2^{-t/30}\right) \Rightarrow y - 20 = C_3\left(2^{-t/30}\right)$. Now $y(30) = 60 \Rightarrow 60 - 20 = C_3\left(2^{-30/30}\right) = C_3/2$, $40 = C_3/2 \Rightarrow C_3 = 80$. Therefore, $y = 80\left(2^{-t/30}\right) + 20$.

Now $40 = 80\left(2^{-t/30}\right) + 20 \Rightarrow 20 = 80\left(2^{-t/30}\right) \Rightarrow \frac{1}{4} = 2^{-t/30} \Rightarrow \ln\left(\frac{1}{4}\right) = -\frac{t}{30}\ln 2 \Rightarrow$ $t = -30\frac{\ln\left(\frac{1}{4}\right)}{\ln 2} = -30(-2) = 60$.

6. a. Separating variables, we have $\frac{dH}{H} = -\frac{1}{EBV}dV_L$, implying $\ln|H| = -\frac{1}{EBV}V_L + K$, or $H = Ce^{-\frac{V_L}{EBV}}$.

b. Given $EBV = 5$ and $H(0) = 0.40$, we can write the solution given in part (a) as $H = 0.4e^{-\frac{V_L}{5}}$. Thus the patient's *volume* of red blood cells at the end of the operation is given by $5H = 2e^{-\frac{V_L}{5}}$. For example, if the patient loses 2.5 liters of blood during surgery ($V_L = 2.5$), then the patient's volume of red blood cells at the end of the operation is $2e^{-0.5} = 1.213$ liters.

7. $\frac{dV}{dh} = 16\sqrt{4-(h-2)^2} \Rightarrow \int dV = \int 16\sqrt{4-(h-2)^2}dh = 16\int\sqrt{4-(h-2)^2}dh$. Now let $h-2 = 2\sin\theta$, so that $dh = 2\cos\theta\, d\theta$. Then we have $V = 16\int\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\, d\theta = 16\int 2\cos\theta\cdot 2\cos\theta\, d\theta = 64\int\cos^2\theta\, d\theta = 64\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C = 32\theta + 16\sin 2\theta + C$. Instead of converting back to the variable h, just notice that $h = 0$ corresponds to $\sin\theta = -1$, or $\theta = -\pi/2$, and $h = 4$ corresponds to $\sin\theta = 1$, or $\theta = \pi/2$. Now use the initial condition to find C: $0 = V(0) = 32(-\pi/2) + 16\sin(-\pi) + C$, or $C = 16\pi$. Therefore $V = 32\theta + 16\sin 2\theta + 16\pi$ and $V(4) = 32(\pi/2) + 16\sin\pi + 16\pi = 32\pi$ units.

8. Separating variables and integrating, we get $\int\frac{dm}{\sqrt{1+m^2}} = \int dx = x + C$. You can attempt to evaluate the first integral by starting with the substitution $m = \tan u$, so that $\sqrt{1+m^2} = \sec u$ and $dm = \sec^2 u\, du$—or you can consult a table of integrals to find that $\int\frac{dm}{\sqrt{1+m^2}} = \ln\left(m + \sqrt{1+m^2}\right) + K$. (We don't need an absolute value inside the logarithm because $\sqrt{1+m^2} > \sqrt{m^2} = |m|$, so that $m + \sqrt{1+m^2} > 0$.) Now we have the equation $\ln\left(m + \sqrt{1+m^2}\right) = x + C$. Using the given initial condition $m(0) = 0$, we find that $\ln\left(0 + \sqrt{1+0^2}\right) = 0 + C$, so that $C = 0$. Now $\ln\left(m + \sqrt{1+m^2}\right) = x \Rightarrow m + \sqrt{1+m^2} = e^x$. Replacing x by $-x$, we see that $-m + \sqrt{1+m^2} = e^{-x}$, so that subtracting the second formula from the first gives us $2m = e^x - e^{-x}$, or $m = \left(e^x - e^{-x}\right)/2 = \sinh(x)$, the *hyperbolic sine* of x.

9. a. We have $x(t) = \frac{\alpha\beta\left(1-e^{(\alpha-\beta)kt}\right)}{\beta-\alpha e^{(\alpha-\beta)kt}} = \frac{e^{(\alpha-\beta)kt}\left(\frac{\alpha\beta}{e^{(\alpha-\beta)kt}}-\alpha\beta\right)}{e^{(\alpha-\beta)kt}\left(\frac{\beta}{e^{(\alpha-\beta)kt}}-\alpha\right)} = \frac{\left(\frac{\alpha\beta}{e^{(\alpha-\beta)kt}}-\alpha\beta\right)}{\left(\frac{\beta}{e^{(\alpha-\beta)kt}}-\alpha\right)} \to \frac{0-\alpha\beta}{0-\alpha} = \beta$. (If $\alpha > \beta$, then $\alpha - \beta > 0$ and $1/e^{(\alpha-\beta)kt} \to 0$ as $t \to \infty$.)

b. If $\alpha < \beta$, then $\alpha - \beta < 0$ and $e^{(\alpha-\beta)kt} \to 0$ as $t \to \infty$. Therefore, $x(t) = \frac{\alpha\beta\left(1-e^{(\alpha-\beta)kt}\right)}{\beta-\alpha e^{(\alpha-\beta)kt}} \to \frac{\alpha\beta(1-0)}{\beta-\alpha\cdot 0} = \frac{\alpha\beta}{\beta} = \alpha$ as $t \to \infty$.

10. Separating variables and integrating, we get $\int\frac{dQ}{Q(Q^2+2)} = \int\frac{dt}{t(t+3)}$, or $\frac{1}{2}\int\left(\frac{1}{Q} - \frac{Q}{Q^2+2}\right)dQ = \frac{1}{3}\int\left(\frac{1}{t} - \frac{1}{t+3}\right)dt$, so that $\frac{1}{2}\left(\ln|Q| - \frac{1}{2}\ln\left|Q^2+2\right|\right) = \frac{1}{3}\left(\ln|t| - \ln|t+3|\right) + C_1$. Multiplying by 4 and simplifying, we get $\ln\left|\frac{Q^2}{Q^2+2}\right| = \frac{4}{3}\ln\left|\frac{t}{t+3}\right| + C_2$, leading to $\frac{Q^2}{Q^2+2} = C\left(\frac{t}{t+3}\right)^{\frac{4}{3}}$. Now the initial condition $Q(1) = 1$

gives us $\frac{1}{1+2} = C\left(\frac{1}{1+3}\right)^{\frac{4}{3}}$, or $C = 4^{\frac{4}{3}}/3$. Therefore $\frac{Q^2}{Q^2+2} = \frac{4^{\frac{4}{3}}}{3}\left(\frac{t}{t+3}\right)^{\frac{4}{3}} = \frac{1}{3}\left(\frac{4t}{t+3}\right)^{\frac{4}{3}}$ and we can solve for Q as an explicit function of t: $Q(t) = \sqrt{\frac{2\left(\frac{4t}{t+3}\right)^{\frac{4}{3}}}{3-\left(\frac{4t}{t+3}\right)^{\frac{4}{3}}}} = \sqrt{\frac{2}{3\left(\frac{t+3}{4t}\right)^{\frac{4}{3}}-1}}$, where we have chosen the positive square root to be consistent with the initial condition.

C

1. Let $v = v(t)$ denote the velocity of the bullet at time t. Let V be the velocity of the bullet at impact, D the final depth of penetration into the bale, and T the time required for achieving D. We have $\frac{dv}{dt} = -k\sqrt{v}$, where k is a positive constant of proportionality (the "coefficient of friction"). We solve this separable equation to find that $v = \frac{(C-kt)^2}{4}$, where C is an arbitrary constant. Assuming that $v(0) = V$ and $v(T) = 0$ (*boundary conditions*), we can determine the constants k and C: $k = 2\sqrt{V}/T$ and $C = 2\sqrt{V}$. Therefore, $v = \left(V/T^2\right)(T-t)^2$. If we let x denote the distance traveled by the bullet in the bale of cotton, then $v = \frac{dx}{dt}$ and we can integrate to get $x = \int \frac{dx}{dt}dt = \int v(t)dt = \int \left(V/T^2\right)(T-t)^2 dt = \left(V/T^2\right)\int (T-t)^2 dt = -\left(V/3T^2\right)(T-t)^3 + C^*$. Because $x = 0$ when $t = 0$, we find that $C^* = VT/3$. Also notice that, because $x(T) = D$, we have $C^* = D(= VT/3)$. In our problem, $T = 0.1$ and $D = 10$. Therefore, $V = 3D/T = 300$ ft./sec.

2. $\frac{dL}{dt} = \frac{aL^n}{b+L^n} \Rightarrow \int \left(\frac{b+L^n}{aL^n}\right)dL = \int dt \Rightarrow \int \left(\frac{b}{a}L^{-n} + \frac{1}{a}\right)dL = \int dt \Rightarrow \frac{b}{a}\left(\frac{L^{-n+1}}{-n+1}\right) + \frac{L}{a} = t + C$. Since we can assume that $L = 0$ when $t = 0$, we see that $C = 0$. Therefore, we can write $t = \frac{1}{a}\left(L + \frac{bL^{1-n}}{1-n}\right)$.

3. **a.** If we let $p(x) = dy/dx$, then $dp/dx = d^2y/dx^2$ and the original equation becomes $\frac{dp}{dx} = k\left[1+p^2\right]^{\frac{1}{2}}$.

b. Then $\frac{dp}{\sqrt{1+p^2}} = k\,dx$, $\int \frac{dp}{\sqrt{1+p^2}} = kx + C_1$, $\ln\left(p + \sqrt{1+p^2}\right) = kx + C_1$, $p + \sqrt{1+p^2} = C_2e^{kx}$. Isolating the radical, squaring both sides, and solving for p, we find that $p = \frac{dy}{dx} = \frac{C_2}{2}e^{kx} - \frac{1}{2C_2}e^{-kx}$.

Integrating with respect to x, we get $y = \frac{C_2}{2k}e^{kx} + \frac{1}{2kC_2}e^{-kx} + C_3 = \frac{1}{2k}\left(C_2e^{kx} + \frac{1}{C_2}e^{-kx}\right) + C_3$. Note that since the original equation is of second-order, we get *two* arbitrary constants in our solution. Also, if we had initial conditions or boundary conditions that would allow us to conclude that $C_2 = 1/C_2 = 1$ and $C_3 = 0$, the resulting solution curve would be given by $y = \frac{1}{k}\cosh(kx)$, a *hyperbolic cosine* whose graph is called a *catenary*.

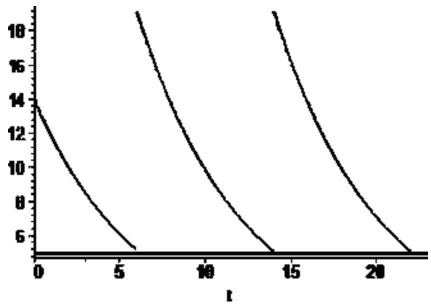
4. **a.** This is a separable equation: $\frac{dC}{dt} = -\frac{C}{6} \Rightarrow \int \frac{dC}{C} = -\frac{1}{6}\int dt \Rightarrow \ln|C| = -\frac{t}{6} + K_1 \Rightarrow |C| = K_2e^{\frac{-t}{6}} \Rightarrow C = Ke^{\frac{-t}{6}}$. Since we are told that $C = 14$ mg/liter at $t = 0$, we can deduce that $K = 14$, so that the concentration at time t can be expressed as $C = C(t) = 14e^{\frac{-t}{6}}$.

b. The concentration becomes ineffective when $C < 5$—that is, when $14e^{\frac{-t}{6}} < 5$. Now $14e^{\frac{-t}{6}} < 5 \Rightarrow e^{\frac{-t}{6}} < 5/14 \Rightarrow -t/6 < \ln(5/14) \Rightarrow t > -6\ln(5/14) \approx 6.18$ hours. Therefore a second injection should be given after about 6 hours.

c. Mathematically, the fact that a second injection increases the concentration by 14 mg/liter means that we have a new initial condition: $C(6) = 14e^{-6/6} + 14 = 14e^{-1} + 14 =$ the concentration 6 hours after the first injection + the increase due to the second injection. This says that the concentration t hours after the *second* injection can be expressed as $C = (14e^{-1} + 14)e^{\frac{-t}{6}}$. Now the concentration becomes ineffective when $\left(14e^{-1} + 14\right)e^{\frac{-t}{6}} < 5$, which implies that $t > -6\ln\left(5/\left(14e^{-1} + 14\right)\right) \approx 8.06$ hours. So another injection is necessary about 8 hours after the second injection.

d. Undesirable side effects occur when the concentration exceeds 20 mg/liter. This translates into $14e^{\frac{-t}{6}} + 14 > 20$ for the second injection. Solving the inequality gives us $t < -6\ln(3/7) \approx 5.08$ hours, which means that we should wait at least 5 hours before giving the second injection. The results of (b) and (d) say that there is an optimal "window" between 5 and 6 hours during which the first injection is still effective but a second injection won't be harmful.

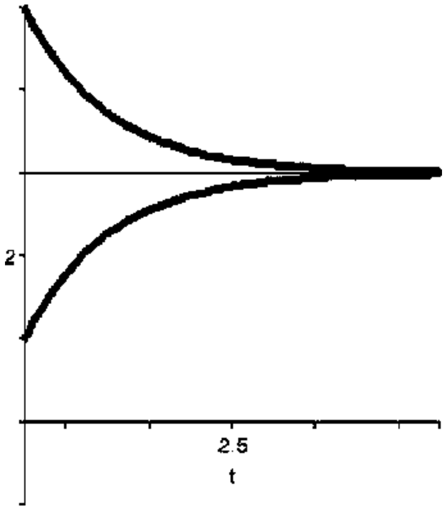
e.



5. **a.** The equilibrium solution occurs when $-\mu C + D = 0$—that is, when $C = \frac{D}{\mu}$.

b. Separating variables, we find that $\int \frac{dC}{-\mu C + D} = \int dt$, which implies that $-\frac{1}{\mu}\ln|D - \mu C| = t + K_1, \ln|D - \mu C| = -\mu t + K_2, |D - \mu C| = K_3 e^{-\mu t}, D - \mu C = Ke^{-\mu t}$, so that $C = \frac{D}{\mu} - \left(\frac{K}{\mu}\right)e^{-\mu t}$. The initial condition $C = C_0$ when $t = 0$ gives us $C_0 = C(0) = \frac{D}{\mu} - \left(\frac{K}{\mu}\right)e^0 = \frac{D-K}{\mu}$, allowing us to conclude that $K = D - \mu C_0$. The final formula for the concentration is $C = \frac{D}{\mu} - \left(\frac{D-\mu C_0}{\mu}\right)e^{-\mu t}$. As $t \to \infty, e^{-\mu t} \to 0$, so that $C(t) \to \frac{D}{\mu} - 0 = \frac{D}{\mu}$, the equilibrium solution found in part (a).

c.



6. a. After separating variables, we have $\int \frac{dP}{P(1-P)} = \int dt \Rightarrow \int\left(\frac{1}{1-P} + \frac{1}{P}\right)dP = \int dt \Rightarrow -\ln|1-P| + \ln|P| = t + C_1 \Rightarrow \ln\left|\frac{P}{1-P}\right| = t + C_1 \Rightarrow \left|\frac{P}{1-P}\right| = C_2e^t \Rightarrow \frac{P}{1-P} = Ce^t \Rightarrow P = \frac{Ce^t}{1+Ce^t}$. You should note that $P \equiv 0$ and $P \equiv 1$ are *equilibrium solutions* (see part (a) of Exercise C5), with $P \equiv 1$ a **singular solution**.

b. $P_0 = P(0) = C/(1+C)$ (from (a)) $\Rightarrow C = P_0/(1-P_0) \Rightarrow P(t) = \frac{\left(\frac{P_0}{1-P_0}\right)e^t}{1+\left(\frac{P_0}{1-P_0}\right)e^t} = \frac{P_0e^t}{(1-P_0)+P_0e^t} = \frac{e^tP_0}{e^t\left(P_0+\frac{1-P_0}{e^t}\right)} = \frac{P_0}{\left(P_0+\frac{1-P_0}{e^t}\right)} \rightarrow \frac{P_0}{(P_0+0)} = 1$ as $t \rightarrow \infty$.

c. From the last expression in part (b), it is clear that $P(t) \rightarrow 1$ as $t \rightarrow \infty$, whether P_0 is between 0 and 1 or is greater than 1. The only difference between the two cases is that when $0 < P_0 < 1, P(t) \rightarrow 1$ from *below* as $t \rightarrow \infty$; while for $P_0 > 1, P(t) \rightarrow 1$ from *above* as $t \rightarrow \infty$. The equilibrium solution $P \equiv 1$ is a **sink**.

2.2 LINEAR EQUATIONS

According to Ince [*Ordinary Differential Equations* (Dover, 1956)], investigations of linear equations were underway before 1700. Ince attributes the development of the concept of the integrating factor to Euler, even though the method had been applied by others earlier. The general treatment of homogeneous linear equations with constant coefficients was begun (according to Ince) with a 1739 letter from Euler to John Bernoulli.

Usually I treat the concept of linearity and the Superposition Principle lightly the first time around, making connections with the familiar processes (transformations) of differentiation and integration. The Superposition Principle will be discussed further in Chapters 4 and 5.

The ideas presented in Exercise C3 are important. Alternatively, we can use the fact (already stated in the text) that a first-order homogeneous linear equation is separable. After finding the general solution $y(x)$ of the homogeneous equation, suggest that a nonzero forcing term $Q(x)$ may be "reached" by considering a solution of the form $y^* = u(x) \cdot y(x)$. Having the student substitute this function in the nonhomogeneous equation and determine the function $u(x)$ is a good way to prepare him or her for the method of *variation of parameters* presented in Section 4.4.

*The hint given for Exercise B6 should be deleted.

A

1. $y' + 2y = 4x \Rightarrow$ integrating factor $\mu(x) = e^{\int 2dx} = e^{2x}$. Then $e^{2x}(y' + 2y) = e^{2x} \cdot 4x \Rightarrow e^{2x}y' + 2e^{2x}y = e^{2x}4x \Rightarrow \frac{d}{dx}[e^{2x}y] = 4xe^{2x} \Rightarrow e^{2x}y = (2x-1)e^{2x} + C \Rightarrow y = 2x - 1 + Ce^{-2x}$. Note that the solution curves are asymptotic to the line $y = 2x - 1$ as t tends to infinity.
2. $y' + 2xy = xe^{-x^2} \Rightarrow \mu(x) = e^{\int 2x\,dx} = e^{x^2}$. Multiplying each side of the original equation by $\mu(x)$, we get $\frac{d}{dx}\left[e^{x^2}y\right] = x$, which implies that $e^{x^2}y = \frac{x^2}{2} + C$, or $y = \frac{x^2e^{-x^2}}{2} + Ce^{-x^2} = \left(\frac{x^2}{2} + C\right)e^{-x^2}$.
3. $\dot{x} + 2tx = t^3 \Rightarrow \mu(t) = e^{\int 2t\,dt} = e^{t^2} \Rightarrow \frac{d}{dt}\left[e^{t^2}x\right] = t^3e^{t^2} \Rightarrow e^{t^2}x = \int t^3e^{t^2}dt = (t^2-1)\frac{e^{t^2}}{2} + C \Rightarrow x = \frac{t^2}{2} - \frac{1}{2} + Ce^{-t^2}$.
4. $y' + y = \cos x \Rightarrow \mu(x) = e^{\int 1dx} = e^x \Rightarrow \frac{d}{dx}[e^xy] = e^x\cos x \Rightarrow e^xy = \frac{e^x}{2}(\sin x + \cos x) + C \Rightarrow y = \frac{1}{2}(\sin x + \cos x) + Ce^{-x}$.
5. $ty' = -3y + t^3 - t^2 \Rightarrow ty' + 3y = t^3 - t^2 \Rightarrow y' + \left(\frac{3}{t}\right)y = t^2 - t \Rightarrow \mu(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = t^3 \Rightarrow \frac{d}{dt}[t^3y] = t^5 - t^4 \Rightarrow t^3y = \frac{t^6}{6} - \frac{t^5}{5} + C \Rightarrow y = \frac{t^3}{6} - \frac{t^2}{5} + \frac{C}{t^3}$.
6. $\frac{dx}{ds} = \frac{x}{s} - s^2 \Rightarrow \frac{dx}{ds} + \left(\frac{-1}{s}\right)x = -s^2 \Rightarrow \mu(s) = e^{\int -\frac{1}{s}ds} = 1/s \Rightarrow \frac{d}{ds}\left[\frac{x}{s}\right] = -s \Rightarrow \frac{x}{s} = -\int s\,ds = -s^2/2 + C \Rightarrow x = -s^3/2 + Cs$.
7. $y = x(y' - x\cos x) = xy' - x^2\cos x \Rightarrow xy' - y = x^2\cos x \Rightarrow y' + \left(\frac{-1}{x}\right)y = x\cos x \Rightarrow \mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = 1/x \Rightarrow \frac{d}{dx}\left[\frac{y}{x}\right] = \cos x \Rightarrow \frac{y}{x} = \int \cos x\,dx = \sin x + C \Rightarrow y = x\sin x + Cx$.
8. $(1+x^2)y' - 2xy = (1+x^2)^2 \Rightarrow y' + \left(\frac{-2x}{1+x^2}\right)y = 1 + x^2 \Rightarrow \mu(x) = e^{\int -\left(\frac{2x}{1+x^2}\right)dx} = e^{-\ln(1+x^2)} = 1/(1+x^2) \Rightarrow \frac{d}{dx}\left[\frac{y}{1+x^2}\right] = 1 \Rightarrow \frac{y}{1+x^2} = x + C \Rightarrow y = (x+C)(1+x^2)$.
9. $t(x' - x) = (1+t^2)e^t \Rightarrow x' - x = \left(\frac{1+t^2}{t}\right)e^t \Rightarrow \mu(t) = e^{\int -1dt} = e^{-t} \Rightarrow \frac{d}{dt}[e^{-t}x] = \left(\frac{1+t^2}{t}\right) = \frac{1}{t} + t \Rightarrow e^{-t}x = \int\left(\frac{1}{t} + t\right)dt = \ln|t| + t^2/2 + C \Rightarrow x = e^t\left(\ln|t| + t^2/2 + C\right)$.
10. $Q' - (\tan t)Q = \sec t \Rightarrow \mu(t) = e^{\int -\tan t\,dt} = e^{\ln(\cos t)} = \cos t \Rightarrow \frac{d}{dt}[Q\cos t] = 1 \Rightarrow Q\cos t = t + C \Rightarrow Q = \frac{t+C}{\cos t}$. The condition $Q(0) = 0$ implies that $0 = \frac{0+C}{\cos(0)} = C/1 = C$, so that our final answer is $Q(t) = \frac{t}{\cos t} = t\sec t$.

11. $xy' + y - e^x = 0 \Rightarrow xy' + y = e^x \Rightarrow y' + \left(\frac{1}{x}\right)y = \frac{e^x}{x} \Rightarrow \mu(x) = e^{\int \frac{1}{x}dx} = x \Rightarrow \frac{d}{dx}[xy] = e^x \Rightarrow xy = \int e^x dx = e^x + C \Rightarrow y = \frac{e^x + C}{x}$. The initial condition $y(a) = b$ implies that $b = \frac{e^a + C}{a}$, so that $C = ab - e^a$ and we have $y(x) = \frac{e^x + ab - e^a}{x}$. (In the initial condition, clearly a shouldn't be 0.)

12. $(xy' - 1)\ln x = 2y \Rightarrow (x\ln x)y' - 2y = \ln x \Rightarrow y' + \left(\frac{-2}{x\ln x}\right)y = \frac{1}{x} \Rightarrow \mu(x) = e^{\int \frac{-2}{x\ln x}dx} = e^{-2\int \frac{\frac{1}{x}}{\ln x}dx} = e^{-2\ln(|\ln x|)} = \frac{1}{\ln^2 x} \Rightarrow \frac{d}{dx}\left[\frac{y}{\ln^2 x}\right] = \frac{1}{x\ln^2 x} \Rightarrow \frac{y}{\ln^2 x} = \int \frac{1}{x\ln^2 x}dx =$ [letting $u = \ln x$, $du = (1/x)dx$] $- \frac{1}{\ln x} + C \Rightarrow y = C\ln^2 x - \ln x$.

13. $y' + ay = e^{mx} \Rightarrow \mu(x) = e^{\int a dx} = e^{ax} \Rightarrow \frac{d}{dx}[e^{ax}y] = e^{ax}e^{mx} = e^{(a+m)x} e^{ax}y = \int e^{(a+m)x}dx = \frac{e^{(a+m)x}}{a+m} + C$, if $a + m \neq 0$—that is, if $m \neq -a$. For $m \neq -a$, we have $y = \frac{e^{mx}}{a+m} + Ce^{-ax}$. If $m = -a$, then $e^{ax}y = \int e^{(a-a)x}dx = \int 1dx = x + C$, so that $y = xe^{-ax} + Ce^{-ax} = (x+C)e^{-ax}$. [Note: A CAS that can solve ODEs may miss the need for an analysis of two cases.]

14. $y' + \left(\frac{1-2x}{x^2}\right)y = 1 \Rightarrow \mu(x) = e^{\int\left(\frac{1-2x}{x^2}\right)dx} = e^{\int(x^{-2} - \frac{2}{x})dx} = e^{-x^{-1} - 2\ln|x|} = \frac{1}{x^2}e^{-1/x} \Rightarrow \frac{d}{dx}\left[\frac{1}{x^2}e^{-1/x}y\right] = \frac{1}{x^2}e^{-1/x} \Rightarrow \frac{1}{x^2}e^{-1/x}y = \int \frac{1}{x^2}e^{-1/x}dx =$ [letting $u = -1/x$, etc.] $= e^{-1/x} + C \Rightarrow y = x^2 + Cx^2e^{1/x}$.

15. $tx' - \left(\frac{x}{t+1}\right) = t \Rightarrow x' + \left(\frac{-1}{t(t+1)}\right)x = 1 \Rightarrow \mu(t) = e^{\int \frac{-1}{t(t+1)}dt} = e^{\int\left(\frac{1}{t+1} - \frac{1}{t}\right)dt} = e^{\ln|t+1| - \ln|t|} = e^{\ln\left|\frac{t+1}{t}\right|} = \frac{t+1}{t} \Rightarrow \frac{d}{dt}\left[\frac{t+1}{t}x\right] = \frac{t+1}{t} \Rightarrow \frac{t+1}{t}x = \int \frac{t+1}{t}dt = t + \ln|t| + C \Rightarrow x = \left(\frac{t}{t+1}\right)\cdot(t + \ln|t| + C)$. Now $x(1) = 0$ implies that $0 = \left(\frac{1}{1+1}\right)(1 + \ln 1 + C)$, so that $C = -1$ and $x(t) = \left(\frac{t}{t+1}\right)(t + \ln|t| - 1)$.

16. The equation $y = (2x + y^3)y'$ is nonlinear in y. But if we think of y as the independent variable and x as the dependent variable, we can write $y = (2x + y^3)\frac{dy}{dx}, ydx = (2x + y^3)dy, y\frac{dx}{dy} = 2x + y^3, y\frac{dx}{dy} - 2x = y^3, \frac{dx}{dy} + \left(\frac{-2}{y}\right)x = y^2$, which is a linear equation in x. Then $\mu(y) = e^{\int -\frac{2}{y}dy} = \frac{1}{y^2} \Rightarrow \frac{d}{dy}\left[\frac{x}{y^2}\right] = 1 \Rightarrow \frac{x}{y^2} = \int 1dy = y + C \Rightarrow x = y^3 + Cy^2$.

17. $x(e^y - y') = 2$: Unlike the situation in Exercise 16, just switching the roles of y and x doesn't work. The resulting equation would not be linear in either x or y. Noticing that $(e^{-y})' = -y'e^{-y}$, we can multiply each side of the differential equation by e^{-y} to get $x(1 - e^{-y}y') = 2e^{-y}$. Making the substitution $z = e^{-y}$ gives us $x\left(1 + \frac{dz}{dx}\right) = 2z, x + x\frac{dz}{dx} = 2z, x\frac{dz}{dx} - 2z = -x$, and $\frac{dz}{dx} + \left(\frac{-2}{x}\right)z = -1$, *a linear equation in z*. An integrating factor is $e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$. Therefore we have $\frac{d}{dx}\left[\frac{z}{x^2}\right] = -\frac{1}{x^2} \Rightarrow \frac{z}{x^2} = \int -\frac{1}{x^2}dx = \frac{1}{x} + C \Rightarrow z = x + Cx^2$, or $e^{-y} = x + Cx^2 \Rightarrow y = -\ln(x + Cx^2)$.

18. $y(x) = \int_0^x y(t)dt + x + 1 \Rightarrow \frac{dy}{dx} = y(x) + 1 \Rightarrow \frac{dy}{dx} - y = 1 \Rightarrow \mu(x) = e^{\int -1dx} = e^{-x} \Rightarrow \frac{d}{dx}[e^{-x}y] = e^{-x} \Rightarrow e^{-x}y = \int e^{-x}dx = -e^{-x} + C \Rightarrow y = Ce^x - 1$. *But there is an implied initial condition here*: $y(0) = 1$. (See formula (1.2.1) in Chapter 1.) Therefore, $1 = y(0) = Ce^0 - 1 \Rightarrow C = 2$, so that the solution is $y = 2e^x - 1$. [*Try any other value of* C *and see that the function doesn't satisfy the original equation.*]

B

1. $y' = \frac{4}{t}y - 6ty^2$, or $y' + \left(-\frac{4}{t}\right)y = -6ty^2$: Here $n = 2$. Divide both sides by y^2 to get $y^{-2}y' + \left(\frac{-4}{t}\right)y^{-1} = -6t$. Letting $z = y^{1-2} = y^{-1}$, we have $\frac{dz}{dt} = -y^{-2} \cdot \frac{dy}{dt}$, so that the equation becomes $-\frac{dz}{dt} + \left(-\frac{4}{t}\right)z = -6t$, or $\frac{dz}{dt} + \left(\frac{4}{t}\right)z = 6t$, linear in z. Now $\mu(t) = e^{\int \frac{4}{t}dt} = t^4$, so that we have $\frac{d}{dt}\left[t^4 z\right] = 6t^5$, $t^4 z = \int 6t^5 dt = t^6 + C$, $z = \frac{t^6+C}{t^4}$, $y = \frac{1}{z} = \frac{t^4}{t^6+C}$. Note that $y \equiv 0$ is a **singular solution**.
2. $\dot{x} = \frac{1}{t}x + \sqrt{x}$, or $\frac{dx}{dt} + \left(-\frac{1}{t}\right)x = x^{\frac{1}{2}}$. Let $z = x^{1-\frac{1}{2}} = x^{\frac{1}{2}}$. Then $\frac{dz}{dt} = \frac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dt}$ by the Chain Rule. Dividing by $x^{\frac{1}{2}}$, we get $2\frac{dz}{dt} + \left(-\frac{1}{t}\right)z = 1$, or $\frac{dz}{dt} + \left(-\frac{1}{2t}\right)z = \frac{1}{2}$. Now $\mu(t) = e^{\int -\frac{1}{2t}dt} = \frac{1}{t^{-\frac{1}{2}}}$, so that we have $\frac{d}{dt}\left[\frac{z}{\sqrt{t}}\right] = \frac{1}{2\sqrt{t}} = \frac{1}{2}t^{-\frac{1}{2}}$ and $\frac{z}{\sqrt{t}} = \int \frac{1}{2}t^{-\frac{1}{2}}dt = t^{\frac{1}{2}} + C$. Therefore, $z = t + C\sqrt{t}$ and, replacing z by $x^{\frac{1}{2}}$, we get $x^{\frac{1}{2}} = t + C\sqrt{t}$, or $x = \left(t + C\sqrt{t}\right)^2 = t^2 + 2Ct^{\frac{3}{2}} + C^2 t$. Since we divided by $x^{\frac{1}{2}}$, we check and see that $x \equiv 0$ is a **singular solution**.
3. $\frac{dy}{dx} + y = xy^3$: Here $n = 3$, so that we let $z = y^{1-3} = y^{-2}$ and $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$. Dividing both sides of the original equation by y^3, we get $y^{-3}\frac{dy}{dx} + y^{-2} = x$, or $-\frac{1}{2}\frac{dz}{dx} + z = x$, $\frac{dz}{dx} - 2z = -2x$, so that $\mu(x) = e^{\int -2dx} = e^{-2x}$. Then $\frac{d}{dx}\left[e^{-2x}z\right] = -2xe^{-2x}$, or $e^{-2x}z = \int -2xe^{-2x}dx = \frac{1}{2}e^{-2x}(2x+1) + C$, so that $z = \frac{1}{2}(2x+1) + Ce^{2x} = x + \frac{1}{2} + Ce^{2x}$ and $\frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}$, or $y = \frac{\pm 1}{\sqrt{x+\frac{1}{2}+Ce^{2x}}}$. Note that $y \equiv 0$ is a **singular solution**.
4. $y' + xy = \sqrt{y}$: Here $n = 1/2$. Let $z = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$, so that $\frac{dz}{dx} = \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$. Now divide both sides of the original equation by $\sqrt{y}$: $y^{-\frac{1}{2}}\frac{dy}{dx} + xy^{\frac{1}{2}} = 1$, or $2\frac{dz}{dx} + xz = 1$, and $\frac{dz}{dx} + \left(\frac{x}{2}\right)z = \frac{1}{2}$. Now $\mu(x) = e^{\int \frac{x}{2}dx} = e^{\frac{x^2}{4}} \Rightarrow \frac{d}{dx}\left[e^{\frac{x^2}{4}}z\right] = \frac{1}{2}e^{\frac{x^2}{4}} \Rightarrow e^{\frac{x^2}{4}}z = \int \frac{1}{2}e^{\frac{x^2}{4}}dx + C \Rightarrow$ $z = \frac{1}{2}e^{-\frac{x^2}{4}}\int e^{\frac{x^2}{4}}dx + Ce^{-\frac{x^2}{4}}$. Replacing z by $\sqrt{y}$, we get $\sqrt{y} = \frac{1}{2}e^{-\frac{x^2}{4}}\int e^{\frac{x^2}{4}}dx + Ce^{-\frac{x^2}{4}}$, or $y = \frac{1}{4}e^{-\frac{x^2}{2}}\left(\int e^{\frac{x^2}{4}}dx + 2C\right)^2$.
5. $y' = 2ty + ty^2$: Here $n = 2$. Let $z = y^{1-2} = y^{-1}$, so that $z' = -y^{-2}y'$. The original equation becomes $z' + 2tz = -t$. Now $\mu(t) = e^{\int 2t\,dt} = e^{t^2} \Rightarrow \left[e^{t^2}z\right]' = -te^{t^2} \Rightarrow e^{t^2}z = \int -te^{t^2}dt + C \Rightarrow z = e^{-t^2}\int -te^{t^2}dx + Ce^{-t^2} = -\frac{1}{2} + Ce^{-t^2}$. Replacing z by y^{-1}, we get $y^{-1} = -\frac{1}{2} + Ce^{-t^2}$, $y = \frac{1}{-\frac{1}{2}+Ce^{-t^2}} = \frac{2}{-1+2Ce^{-t^2}} = \frac{2e^{t^2}}{2C-e^{t^2}}$.
6. $y' = x^3y^2 + xy$: We can write this as $y' - xy = x^3y^2$. Now divide by y^2 to get $y^{-2}y' - xy^{-1} = x^3$. Let $z = y^{-1}$. Then $z' = -y^{-2}y'$ and the equation becomes $-z' - xz = x^3$, or $z' + xz = -x^3$. Now $\mu(x) = e^{\int x\,dx} = e^{\frac{x^2}{2}}$ and $\left[e^{\frac{x^2}{2}}z\right]' = -x^3e^{\frac{x^2}{2}}$.

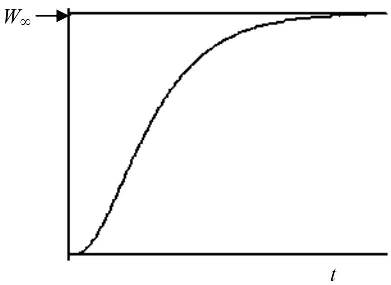
   Thus $e^{\frac{x^2}{2}}z = \int -x^3e^{\frac{x^2}{2}}dx = (2-x^2)e^{\frac{x^2}{2}} + C$, so $z = 2 - x^2 + Ce^{\frac{-x^2}{2}}$. Replacing z by y^{-1}, we get $y^{-1} = 2 - x^2 + Ce^{\frac{-x^2}{2}}$, or $y = \frac{1}{Ce^{\frac{-x^2}{2}} - x^2 + 2} = \frac{e^{\frac{x^2}{2}}}{(2-x^2)e^{\frac{x^2}{2}} + C}$.
7. a. $\frac{dW}{dt} = \alpha W^{\frac{2}{3}} - \beta W$, $\frac{dW}{dt} + \beta W = \alpha W^{\frac{2}{3}}$: Here $n = 2/3$. Letting $z = W^{1-\frac{2}{3}} = W^{\frac{1}{3}}$ and dividing both sides of the ODE by $W^{\frac{2}{3}}$ gives us the new equation $W^{-\frac{2}{3}}\frac{dW}{dt} +$

$\beta W^{\frac{1}{3}} = \alpha$. Noting that $\frac{dz}{dt} = \frac{1}{3}W^{-\frac{2}{3}} \cdot \frac{dW}{dt}$, we can write this last equation as $3\frac{dz}{dt} + \beta z = \alpha$, or $\frac{dz}{dt} + \left(\frac{\beta}{3}\right)z = \frac{\alpha}{3}$, so that $\mu(t) = e^{\int \frac{\beta}{3}dt} = e^{\frac{\beta t}{3}}$. Now $\frac{d}{dt}\left[e^{\frac{\beta t}{3}}z\right] = \frac{\alpha}{3}e^{\frac{\beta t}{3}} \Rightarrow e^{\frac{\beta t}{3}}z = \int \frac{\alpha}{3}e^{\frac{\beta t}{3}}dt = \frac{\alpha}{\beta}e^{\frac{\beta t}{3}} + C \Rightarrow z = \frac{\alpha}{\beta} + Ce^{-\frac{\beta t}{3}}$. Replacing z by $W^{\frac{1}{3}}$, we find that $W^{\frac{1}{3}} = \frac{\alpha}{\beta} + Ce^{-\frac{\beta t}{3}}$, or $W(t) = \left(\frac{\alpha}{\beta} + Ce^{-\frac{\beta t}{3}}\right)^3$.

b. $W_\infty = \lim_{t\to\infty} W(t) = \lim_{t\to\infty}\left(\frac{\alpha}{\beta} + Ce^{-\frac{\beta t}{3}}\right)^3 = \left(\frac{\alpha}{\beta} + C\cdot \lim_{t\to\infty} e^{-\frac{\beta t}{3}}\right)^3 = \left(\frac{\alpha}{\beta}\right)^3$.

c. $W(0) = 0 \Rightarrow 0 = W(0) = \left(\frac{\alpha}{\beta} + Ce^0\right)^3 = \left(\frac{\alpha}{\beta} + C\right)^3 \Rightarrow C = -\frac{\alpha}{\beta}$. Therefore $W(t) = \left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta}e^{-\frac{\beta t}{3}}\right)^3 = \left(\frac{\alpha}{\beta}\right)^3\left(1 - e^{-\frac{\beta t}{3}}\right)^3 = W_\infty\left(1 - e^{-\frac{\beta t}{3}}\right)^3$.

d.



8. Suppose we have the homogeneous linear equation $a_1(x)\frac{dy}{dx} + a_0(x)y = 0$. Dividing by $a_1(x)$, we get $\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = 0$, $\frac{dy}{dx} = -\frac{a_0(x)}{a_1(x)}y$, $\frac{dy}{y} = -\left(\frac{a_0(x)}{a_1(x)}\right)dx$, so that the equation is separable.

9. **a.** $L\frac{dI}{dt} + RI = E \Rightarrow \frac{dI}{dt} + \left(\frac{R}{L}\right)I = \frac{E}{L} \Rightarrow \mu(t) = e^{\int \frac{R}{L}dt} = e^{\left(\frac{R}{L}\right)t} \Rightarrow \frac{d}{dt}\left[e^{\left(\frac{R}{L}\right)t}I\right] = \frac{E}{L}e^{\left(\frac{R}{L}\right)t} \Rightarrow e^{\left(\frac{R}{L}\right)t}I = \int \frac{E}{L}e^{\left(\frac{R}{L}\right)t}dt = \frac{E}{R}e^{\left(\frac{R}{L}\right)t} + C \Rightarrow I = \frac{E}{R} + Ce^{-\left(\frac{R}{L}\right)t}$. Now the wording of the problem suggests that $I(0) = 0$, so that $0 = I(0) = \frac{E}{R} + Ce^0 \Rightarrow C = -\frac{E}{R}$. Therefore, $I(t) = \frac{E}{R} - \frac{E}{R}e^{-\left(\frac{R}{L}\right)t} = \frac{E}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$.

b. $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{E}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right) = \frac{E}{R}\left(1 - \lim_{t\to\infty} e^{-\left(\frac{R}{L}\right)t}\right) = \frac{E}{R}$.

c. By "final" value, we mean the value E/R found in (b). Now we want $I = \frac{1}{2}\frac{E}{R}$, or $\frac{1}{2}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$, so that $\frac{1}{2} = 1 - e^{-\left(\frac{R}{L}\right)t}$, $e^{-\left(\frac{R}{L}\right)t} = \frac{1}{2}$, $-\frac{R}{L}t = \ln\left(\frac{1}{2}\right) = -\ln 2$, $\frac{R}{L}t = \ln 2$, and $t = \frac{L}{R}\ln 2$.

d. Using the general solution found in part (a), we see that $I(0) = \frac{E}{R} \Rightarrow \frac{E}{R} = \frac{E}{R}\left(1 - Ce^0\right) = \frac{E}{R} + C^*$, implying that $C^* = 0$. Therefore $I(t) \equiv \frac{E}{R}$, which we realize is the *equilibrium solution* (see Exercise C5(a) in Exercises 2.1) of the autonomous equation.

10. **a.** $R\frac{dQ}{dt}+\frac{Q}{C}=E \Rightarrow \frac{dQ}{dt}+\left(\frac{1}{RC}\right)Q=E \Rightarrow \mu(t)=e^{\int \frac{1}{RC}dt}=e^{\frac{t}{RC}} \Rightarrow \frac{d}{dt}\left[e^{\frac{t}{RC}}Q\right]=Ee^{\frac{t}{RC}} \Rightarrow$ $e^{\frac{t}{RC}}Q=\int Ee^{\frac{t}{RC}}dt=ERCe^{\frac{t}{RC}}+K \Rightarrow Q=e^{-\frac{t}{RC}}\left(ERCe^{\frac{t}{RC}}+K\right)=ERC+Ke^{-\frac{t}{RC}}$. Then $0=Q(0)=ERC+Ke^{0}=ERC+K \Rightarrow K=-ERC \Rightarrow Q(t)=ERC\left(1-e^{-\frac{t}{RC}}\right)$.

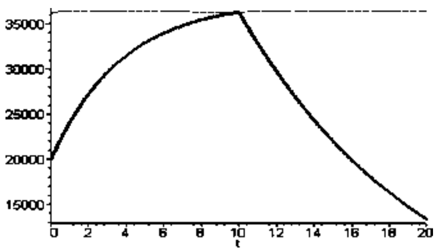
b. The "final" value is calculated as $\lim_{t\to\infty} Q(t)=ERC$. We want $ERC/2=ERC\left(1-e^{-t/RC}\right)$, which implies $e^{-t/RC}=1/2, -t/RC=\ln(1/2)$, and $t=RC\ln 2\approx 0.693RC$.

11. The equation is $R\frac{dQ}{dt}+\frac{Q}{C}=E_0\sin(\omega t)$, with $Q(0)=0$. Then $\frac{dQ}{dt}+\left(\frac{1}{RC}\right)Q=\frac{E_0}{R}\sin(\omega t) \Rightarrow$ $\mu(t)=e^{\int\frac{1}{RC}dt}=e^{\frac{t}{RC}} \Rightarrow \frac{d}{dt}\left[e^{\frac{t}{RC}}Q\right]=\frac{E_0}{R}e^{\frac{t}{RC}}\sin(\omega t) \Rightarrow e^{\frac{t}{RC}}Q=\frac{E_0}{R}\int e^{\frac{t}{RC}}\sin(\omega t)dt=$ $\frac{E_0}{R}\left\{\frac{RCe^{\frac{t}{RC}}[\sin(\omega t)-\omega RC\cos(\omega t)]}{1+(RC\omega)^2}\right\}+K$, so that $Q(t)=\left\{\frac{E_0C[\sin(\omega t)-\omega RC\cos(\omega t)]}{1+(RC\omega)^2}\right\}+Ke^{-\frac{t}{RC}}$. Now $0=Q(0)=\frac{E_0C[0-\omega RC]}{1+(RC\omega)2}+K=-\frac{\omega E_0RC^2}{1+(RC\omega)^2}+K$, so that $K=\frac{\omega E_0RC^2}{1+(RC\omega)^2}$ and $Q(t)=$ $\frac{E_0C[\sin(\omega t)-\omega RC\cos(\omega t)]}{1+(RC\omega)^2}+\frac{\omega E_0RC^2}{1+(RC\omega)^2}e^{-\frac{t}{RC}}=\frac{E_0C}{1+(RC\omega)^2}\left\{\sin(\omega t)-\omega RC\cos(\omega t)+\omega RCe^{-\frac{t}{RC}}\right\}$.

12. **a.** For $0<t<T$, the equation is $\frac{dS}{dt}+\left(\frac{r\bar{A}}{M}+\lambda\right)S=r\bar{A}$. Let $b=\frac{r\bar{A}}{M}+\lambda$. Then $\mu(t)=$ $e^{\int b\,dt}=e^{bt}$ and we have $\frac{d}{dt}\left[e^{bt}S\right]=e^{bt}r\bar{A} \Rightarrow e^{bt}S=r\bar{A}\int e^{bt}\,dt=\frac{r\bar{A}}{b}e^{bt}+C \Rightarrow$ $S(t)=\frac{r\bar{A}}{b}+Ce^{-bt}$ for $0<t<T$. Letting $S_0=S(0)$, we get $S_0=\frac{r\bar{A}}{b}+C$, so that $C=S_0-\frac{r\bar{A}}{b}$ and $S(t)=\frac{r\bar{A}}{b}+\left(S_0-\frac{r\bar{A}}{b}\right)e^{-bt}$ (*) for $0<t<T$. Now for $t>T, A=0$ and the equation becomes $\frac{dS}{dt}+\lambda S=0$, or $\frac{dS}{dt}=-\lambda S$, which is separable and has solution $S=ke^{-\lambda t}$. At $t=T$, let $S=S_T$, so that $S_T=ke^{-\lambda t}$ (**). Hence for $t\geq T, S(t)=S_Te^{-\lambda(t-T)}$. From (*) we see that S_T has the value $\frac{r\bar{A}}{b}+\left(S_0-\frac{r\bar{A}}{b}\right)e^{-bT}$. Combining (*) and (**) and substituting for b, we get the formula for predicted sales:

$$S(t)=\begin{cases}\dfrac{r\bar{A}}{\left(\frac{r\bar{A}}{M}+\lambda\right)}+\left(S_0-\dfrac{r\bar{A}}{\left(\frac{r\bar{A}}{M}+\lambda\right)}\right)e^{-\left(\frac{r\bar{A}}{M}+\lambda\right)t} & \text{for } 0<t<T \\ S_Te^{-\lambda(t-T)} & \text{for } t\geq T\end{cases}.$$

b. Choosing $\bar{A}=1000, r=10, \lambda=0.1, S_0=20000, S_T=36000, M=60000$, and $T=10$, we have the following graph:



13. **a.** $\frac{dp}{dt}=\nu-(\mu+\nu)p$, or $\frac{dp}{dt}+(\mu+\nu)p=\nu$: Here the integrating factor is $e^{\int(\mu+\nu)dt}=e^{(\mu+\nu)t}$, so that $\frac{d}{dt}\left[e^{(\mu+\nu)t}p\right]=\nu e^{(\mu+\nu)t}, e^{(\mu+\nu)t}p=\int \nu e^{(\mu+\nu)t}dt, e^{(\mu+\nu)t}p=\frac{\nu}{\mu+\nu}e^{(\mu+\nu)t}+C$, and $p(t)=\frac{\nu}{\mu+\nu}+Ce^{-(\mu+\nu)t}$. Letting $p_0=p(0)$, we get $p_0=\frac{\nu}{\mu+\nu}+C\cdot e^{0}$, implying that

$C = p_0 - \frac{\nu}{\mu+\nu}$. Therefore $p(t) = \frac{\nu}{\mu+\nu} + \left(p_0 - \frac{\nu}{\mu+\nu}\right)e^{-(\mu+\nu)t} = \frac{\nu}{\mu+\nu}\left[1 - e^{-(\mu+\nu)t}\right] + p_0 e^{-(\mu+\nu)t}$. *Note that* $p(t) \equiv \frac{\nu}{\mu+\nu}$ *is an equilibrium solution: If* $p_0 = \frac{\nu}{\mu+\nu}$, *then* $p(t)$ *stays at this value forever.* Since $q(t) = 1 - p(t)$ and $q_0 = 1 - p_0$, we have $q(t) = 1 - \left\{\frac{\nu}{\mu+\nu}\left[1 - e^{-(\mu+\nu)t}\right] + p_0 e^{-(\mu+\nu)t}\right\} = \frac{\mu}{\mu+\nu} + \left(q_0 - \frac{\mu}{\mu+\nu}\right)e^{-(\mu+\nu)t} = \frac{\mu}{\mu+\nu}\left[1 - e^{-(\mu+\nu)t}\right] + q_0 e^{-(\mu+\nu)t}$.

b. Since $e^{-(\mu+\nu)t} \to 0$ as $t \to \infty$, clearly $p(t) \to \nu/(\mu+\nu)$ and $q(t) \to \mu/(\mu+\nu)$. Notice that both these equilibrium gene frequencies are equilibrium solutions (sinks) for the autonomous differential equation.

14. a. $\frac{dV}{dt} + K(t) = r(t)V \Rightarrow \frac{dV}{dt} - r(t)V = -K(t)$. Ordinarily we would write the integrating factor as $\mu(t) = e^{-\int r(t)dt}$, but note that $\mu(t) = e^{-\int_T^t r(x)dx} = e^{\int_t^T r(x)dx}$ is an integrating factor since $\frac{d}{dt}\mu(t) = -r(t)e^{-\int_T^t r(x)dx} = -r(t)\mu(t)$. Then $\frac{d}{dt}\left[e^{\int_t^T r(x)dx}V\right] = -e^{\int_t^T r(x)dx}K(t) \Rightarrow e^{\int_t^T r(x)dx}V = -\int_T^t K(u)e^{\int_u^T r(x)dx}du + C = \int_t^T K(u)e^{\int_u^T r(x)dx}du + C \Rightarrow V = e^{-\int_t^T r(x)dx} \cdot \int_t^T K(u)e^{\int_u^T r(x)dx}du + Ce^{-\int_t^T r(x)dx}$. Letting $t = T$, the initial condition $V(T) = Z$ implies that $Z = e^0 \cdot 0 + Ce^0$, or $C = Z$. Therefore $V(t) = e^{-\int_t^T r(x)dx} \cdot \int_t^T K(u)e^{\int_u^T r(x)dx}du + Ze^{-\int_t^T r(x)dx} = e^{-\int_t^T r(x)dx}\left(Z + \int_t^T K(u)e^{\int_u^T r(x)dx}du\right)$.

b. If $K(t) \equiv 0$, then $V(t) = Ze^{-\int_t^T r(x)dx}$.

C

1. $y' + a(x)y = b(x)y^n$: Let $y = u^{1/(1-n)}$, $n \neq 0, 1$. Then $y' = \frac{1}{1-n}u^{\frac{1}{1-n}-1}u' = \frac{u^{n/(1-n)}}{1-n}u'$. Substituting, we get $\frac{u^{n/(1-n)}}{1-n}u' + a(x)u^{1/(1-n)} = b(x)u^{n/(1-n)}$. Now divide both sides by $\frac{u^{n/(1-n)}}{1-n}$ to get $u' + (1-n)a(x)u(x) = (1-n)b(x)$, a linear equation.

2. This is a Bernoulli equation. If we divide both sides by I^2, we get $I^{-2}\frac{dI}{dt} - k(P_0 + rt)I^{-1} = -k$. Letting $z = I^{-1}$, we have $\frac{dz}{dt} = -I^{-2}\frac{dI}{dt}$, so our original equation becomes $-\frac{dz}{dt} - k(P_0 + rt)z = -k$, or $\frac{dz}{dt}k(P_0 + rt)z = k$, a linear equation. Then $\mu(t) = e^{\int k(P_0+rt)dt} = e^{kP_0t+(1/2)krt^2}$ and $\frac{d}{dt}\left[e^{\int k(P_0+rt)dt}z\right] = ke^{kP_0t+(1/2)krt^2}$, $e^{kP_0t+(1/2)krt^2}z = k\int e^{kP_0t+(1/2)krt^2}dt + C = k\int_0^t e^{kP_0u+(1/2)kru^2}du + C$, and $z = e^{-[kP_0t+(1/2)krt^2]}\left(k\int_0^t e^{kP_0u+(1/2)kru^2}du + C\right)$. Now $z(0) = I(0)^{-1} = 1/I_0 = e^0(0 + C)$, so $C = 1/I_0$ and $z = e^{-[kP_0t+(1/2)krt^2]}\left(\frac{1}{I_0} + k\int_0^t e^{kP_0u+(1/2)kru^2}du\right)$. Inverting, we find that $I = \frac{1}{z} = e^{kP_0t+(1/2)krt^2}\left(\frac{1}{I_0} + k\int_0^t e^{kP_0u+(1/2)kru^2}du\right)^{-1}$.

3. a. If $Q(x) \equiv 0$, then the equation has the form $\frac{dy}{dx} + P(x)y = 0$, or $\frac{dy}{dx} = -P(x)y$, so that we can separate the variables and get $\frac{dy}{y} = -P(x)dx$. Integrating, we get $\ln|y| = -\int P(x)dx + C_1$, so that $|y| = C_2e^{-\int P(x)dx}$ and $y = Ce^{-\int P(x)dx}$. This is the general solution, y_{GH}, of the homogeneous equation.

b. Letting $y(x) = e^{-\int P(x)dx} \cdot \int e^{\int P(x)dx}Q(x)dx$, we see (Product Rule and FTC) that $dy/dx = e^{-\int P(x)dx} \cdot e^{\int P(x)dx}Q(x) - P(x)e^{-\int P(x)dx} \cdot \int e^{\int P(x)dx}Q(x)dx = Q(x) - P(x)e^{-\int P(x)dx}$.

$\int e^{\int P(x)dx} Q(x)dx$ and $dy/dx + P(x)y = Q(x) - P(x)e^{-\int P(x)dx} \cdot \int e^{\int P(x)dx} Q(x)dx + P(x) \cdot \left(e^{-\int P(x)dx} \int e^{\int P(x)dx} Q(x)dx\right) = Q(x)$, so that $y(x)$ is a particular solution of the nonhomogeneous equation. Thus formula (2.2.2) can be expressed as $y_{\text{GNH}} = y_{\text{GH}} + y_{\text{PNH}}$.

c. Since $L(y) = \frac{dy}{dx} + P(x)y$ is a linear operator, the Superposition Principle yields $L(y_{\text{GNH}}) = L(y_{\text{GH}} + y_{\text{PNH}}) = L(y_{\text{GH}}) + L(y_{\text{PNH}}) = 0 + Q(x) = Q(x)$, as expected.

2.3 COMPARTMENT PROBLEMS

I consider compartment (or mixing) problems important illustrations of the power of linear ODEs. Students often have difficulties with these kinds of "word problems," but usually find them interesting. As is obvious from Examples 2.3.1–2.3.3, I emphasize the units of measurement involved in such problems.

A

1. $\frac{dp(t)}{dt} = bp(t) - dp(t) = \overbrace{(b-d)}^{\text{net growth rate}} p(t)$.
2. Suppose $d = \alpha p(t)$, where α is a constant. Then we have $\frac{dp(t)}{dt} = bp(t) - (\alpha p(t)) \cdot p(t) = p(t)(b - \alpha p(t)) = bp(t)\left(1 - \frac{\alpha}{b}p(t)\right)$.
3. Suppose $b = \beta p(t)$, where β is a constant. Then we have $\frac{dp(t)}{dt} = \beta p(t) \cdot p(t) - dp(t) = p(t)(\beta p(t) - d) = \beta p(t)\left(p(t) - \frac{d}{\beta}\right)$.
4. $p(t) =$ rate of inflow − rate of outflow $= [bp(t) + I(t)] - dp(t) = (b-d)p(t) + I$.

B

1. a. $\frac{dP}{dt} = kP - \alpha t \Rightarrow \frac{dP}{dt} - kP = -\alpha t \Rightarrow \mu(t) = e^{\int -kdt} = e^{-kt} \Rightarrow \frac{d}{dt}\left[e^{-kt}P\right], = -\alpha te^{-kt} \Rightarrow e^{-kt}P = -\alpha \int te^{-kt}dt = -\alpha\left(\frac{-e^{-kt}}{k^2}(1+kt)\right) + C \Rightarrow P = \frac{\alpha t}{k} + \frac{\alpha}{k^2} + Ce^{kt}$. Now $P(0) = 1.285 \Rightarrow 1.285 = \frac{\alpha}{k^2} + C$, or $C = 1.285 - \frac{\alpha}{k^2}$, so that we can write the solution as $P = \frac{\alpha t}{k} + \frac{\alpha}{k^2} + \left(1.285 - \frac{\alpha}{k^2}\right)e^{kt} = \frac{(1.60625 \times 10^{-3})t}{0.0355} + \frac{1.60625 \times 10^{-3}}{(0.0355)^2} + \left(1.285 - \frac{1.60625 \times 10^{-3}}{(0.0355)^2}\right)e^{0.0355t} \approx (0.0452t + 1.275) + 0.01045e^{0.0355t}$.

   b. In the year 2010, $t = 20$. Then $P(20) = \frac{(1.60625 \times 10^{-3})(20)}{0.0355} + \frac{1.60625 \times 10^{-3}}{(0.0355)^2} + \left(1.285 - \frac{1.60625 \times 10^{-3}}{(0.0355)^2}\right) e^{0.0355(20)} = 2.200736130 \approx$ **2,200,736** people.
2. First note that the volume of material in the tank changes with time. Then **net rate = rate of inflow − rate of outflow**, so that the volume of mixture in the tank is increasing at the net rate of 4 gal/min − 3 gal/min = 1 gal/min. This says that $dV/dt = 1$, so that $V(t) = t + C$. Since $V(0) = 50$ gal, we have $V(t) = t + 50$. If we let $Q(t)$ denote the *quantity* of potassium (in grams) in the tank at time t, with $Q(0) = 0$, then the *concentration*

in the tank at time t is $Q/(t+50)$, in units of gm/gal. Therefore we have

$$\underbrace{\frac{dQ}{dt}}_{\text{net rate of change of potassium in tank (gm/min)}} = \underbrace{\left(4\,\frac{\text{gal}}{\text{min}}\cdot 10\,\frac{\text{gm}}{\text{gal}}\right)}_{\text{rate of inflow (gm/min)}} - \underbrace{\left(3\,\frac{\text{gal}}{\text{min}}\cdot \frac{Q(t)}{t+50}\,\frac{\text{gm}}{\text{gal}}\right)}_{\text{rate of outflow (gm/min)}},$$

so that we have $\frac{dQ}{dt} + \left(\frac{3}{t+50}\right)Q = 40$, a linear equation with integrating factor $\mu(t) = e^{\int \frac{3}{t+50}\,dt} = (t+50)^3$. Then $\frac{d}{dt}\left[(t+50)^3 Q\right] = 40(t+50)^3$, so that $(t+50)^3 Q = \int 40(t+50)^3 dt = 10(t+50)^4 + C$, or $Q(t) = \frac{10(t+50)^4+C}{(t+50)^3} = 10(t+50) + \frac{C}{(t+50)^3}$. The initial condition $Q(0) = 0 \Rightarrow 0 = 10(50) + \frac{C}{(50)^3}$, so that $C = -10(50)^4$ and $Q(t) = 10(t+50) - \frac{10(50)^4}{(t+50)^3}$. Since $V(t) = t + 50$ and the tank holds 100 gallons, the tank will be full when $t = 50$. Then $Q(50) = 10(50+50) - \frac{10(50)^4}{(50+50)^3} = 937.5$ gms and the concentration of potassium at this time is $\frac{937.5 \text{ gms}}{100 \text{ gal}} = 9.375$ gms/gal.

3. Let $A(t)$ denote the amount of salt in the tank after t minutes. Since the tank is initially full of pure water, $A(0) = 0$. The rate at which salt is being added to the tank is $\left(\frac{1}{4}\text{ lb/gal}\right)(1\text{ gal/min}) = \frac{1}{4}$ lb/min. The rate (2 gal/min) at which both pure water and brine are entering the tank is equal to the rate at which the mixture is being removed, so that at any time t the amount of liquid in the tank is constant at 100 gallons. Then the *concentration* of salt in solution in the tank is expressed as $A(t)/100$. Consequently, the rate at which salt is being removed from the tank is then $\left(\frac{A(t)}{100}\,\frac{\text{lb}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = \frac{A(t)}{50}\,\frac{\text{lb}}{\text{min}}$. Since **net rate = rate of inflow – rate of outflow**, we have the IVP $\frac{dA}{dt} = \frac{1}{4} - \frac{A(t)}{50}$, $A(0) = 0$. We can write the equation in the standard form of a linear equation: $\frac{dA}{dt} + \frac{1}{50}A(t) = \frac{1}{4}$, which has an integrating factor $\mu(t) = e^{\int \frac{1}{50}dt} = e^{\frac{t}{50}}$. Then $\frac{d}{dt}\left[e^{\frac{t}{50}}A\right] = \frac{1}{4}e^{\frac{t}{50}}$, so that $e^{\frac{t}{50}}A = \frac{1}{4}\int e^{\frac{t}{50}}dt = \frac{25}{2}e^{\frac{t}{50}} + C$, or $A(t) = \frac{25}{2} + Ce^{-\frac{t}{50}}$. The initial condition $A(0) = 0$ implies that $0 = A(0) = \frac{25}{2} + C$, so that $C = -25/2$ and $A(t) = \frac{25}{2} - \frac{25}{2}e^{-\frac{t}{50}} = \frac{25}{2}\left(1 - e^{-\frac{t}{50}}\right)$. [Note: The differential equation can also be solved as a *separable* equation.]

4. **a.** If $X(t)$ is the amount (in grams, for example) of chlorine in solution at time t, then the rate at which chlorine is entering the tank is $(0.01\text{gm/gal})(2\text{gal/sec}) = 2/100\text{gm/sec}$ and the rate at which chlorine is leaving is

$$\left(\frac{X}{200-t}\,\frac{\text{gm}}{\text{gal}}\right)\cdot\left(3\,\frac{\text{gal}}{\text{sec}}\right) = \frac{3X}{200-t}\,\frac{\text{gm}}{\text{sec}}.$$

Note that the net amount of liquid in the tank is changing at the rate 2 gal/sec − 3 gal/sec = −1 gal/sec, so that the amount of liquid in the tank at any time t is given by $200 - t$. Using the principle **net rate = rate of inflow – rate of outflow**, we get the equation

$$\frac{dX}{dt} = \frac{2}{100} - \frac{3X}{200-t}, \quad \text{or} \quad \frac{dX}{dt} + \left(\frac{3}{200-t}\right)X = \frac{2}{100}.$$

Multiplying by the integrating factor $e^{\int \frac{3}{200-t}dt} = e^{-3\ln(200-t)} = (200-t)^{-3}$ and integrating, we get $X(t) = \frac{1}{100}(200-t) + C(200-t)^3$.

Since $X(0) = 0$, we find that $0 = \frac{200}{100} + C(200)^3$, or $C = -\frac{200}{100(200)^3} = -\frac{2}{(200)^3}$. Therefore $X(t) = \frac{1}{100}(200-t) - \frac{2}{(200)^3}(200-t)^3$. Now the tank is half full when $200 - t = 100$, or $t = 100$, so that the concentration of chlorine at this time is $\frac{X(100)}{100} = \frac{100}{10000} - \frac{2(100)^3}{100(200)^3} = 0.0075 = 0.75\%$ solution.

b. When the tank is half full, it contains 100 gallons. If 100 gallons of 1% solution is added, then 1 gram (= 100 gal × 0.01 gm/gal) is added to the $\frac{3}{4}$ gram (= 100 gal × 0.0075 gm/gal), making a total of 1.75 grams of chlorine in the 200 gallon tank, resulting in a concentration of $1.75/200 = 0.00875$ gm/gal $= 0.875\%$.

5. a. Because liquid is running into the tank at the rate of 3 gal/min and running out of the tank at the rate of 2 gal/min, the net effect is that the tank is increasing its liquid content at the rate of 1 gal/min. Thus at time t(in minutes), the tank has $50+t$ gallons of liquid. After 50 minutes, the tank contains $50 + 50 =$ **100** gallons of liquid.

b. The rate of **inflow** of the salt (in pounds per minute) is 2 lbs/gal × 3 gal/min = 6 lbs/min. If $Q(t)$ denotes the quantity (pounds) of salt in the tank at time t, then the rate at which salt is *leaving* the tank (the rate of **outflow**) is given by 2 gal/min × $Q(t)/(50+t)$ lbs/gal $= 2Q(t)/(50+t)$ lbs/min. Thus the differential equation is $\frac{dQ}{dt} = 6 - \frac{2Q}{50+t}$, a linear equation which can be written as $\frac{dQ}{dt} + \left(\frac{2}{50+t}\right)Q = 6$. The integrating factor is $\mu(t) = e^{\int \frac{2}{50+t}dt} = (50+t)^2$. Then $\frac{d}{dt}\left[(50+t)^2 Q\right] = 6(50+t)^2$, so $(50+t)^2 Q = 6\int (50+t)^2 dt = 2(50+t)^3 + C$ and $Q = 2(50+t) + C(50+t)^{-2}$. Since $Q(0) = 0$, we find that $C = -2(50^3)$. When $t = 50$, $Q = 2(50+50) - 2(50^3)(50+50)^{-2} = 200 - 25 = 175$ pounds.

6. Note that the rate of **inflow** of the salt is **0** lbs/min because only fresh (pure) water enters the tank. The volume of fluid in the tank at time t is $(100+t)$ gallons since the net rate of increase of the initial fluid in the tank is $(3-2)$ gallons per minute. Now let $Q(t)$ denote the amount of salt (in pounds) in solution at time t. Then $dQ/dt = -2Q/(100+t)$ since 2 gal/min of the mixture runs out. Separating the variables and integrating, we get $\ln Q = -2\ln(100+t) + C$. Since $Q(0) = 75$, we find that $C = \ln\left[75(100^2)\right]$. Exponentiating, we conclude that $Q(t) = 75(100^2)/(100+t)^2$. When $t = 1.5$ hours $= 90$ minutes, we have $Q \approx 20.8$ lbs.

7. Let $Q(t)$ be the amount of the pollutant in the pool at time t. When $t = 0$, $Q = 20$. We want Q to be less than 1. If we leave the pool filled and flush it with pure water, Then $dQ/dt = -100Q/10^4$, which has the solution $Q = Ce^{-t/100}$. We have $C = 20$, so asking for $Q = 1$ gives us $t \approx 300$ minutes ≈ 5 hours.

If we, instead, empty the pool to the halfway point without adding more water (thereby removing half the pollutant), which takes 50 minutes, and then start the flushing policy, we have the equation $dQ/dt = -100Q/5000$ with initial condition $Q(0) = 10$. This has the solution $Q(t) = 10e^{-t/50}$; and Q reaches 1 when $t \approx 115.13$ minutes. The total time will then be 165.12 minutes, which saves about 2.2 hours over the other method. Refilling would take 50 more minutes.

8. If *all* new employees were women, the differential equation would be $\frac{dW}{dt} = 50 - \left(\frac{100}{6000-50t}\right)W$, which leads to $W(t) = (6000 - 50t) + C(6000 - 50t)^2$. Now $W(t) = 1500$ when $t = 0$, so that $C = -1/8000$. Then when $t = 40$, $W(t) = 2000$, which is one-half, or **50%**, of the staff at that time.

C

1. Let $X(t)$ denote the concentration (in gm/cm^3) of drugs in the organ at time t. Then $X(0) = 0$ and we want $X(t) \le c_{\max}$. The *amount* of the drug entering the organ is given by $\left(r\frac{\text{cm}^3}{\text{sec}}\right) \times \left(c\frac{\text{gm}}{\text{cm}^3}\right) = rc$ gm/sec and the amount leaving is $\left(r\frac{\text{cm}^3}{\text{sec}}\right) \times \left(X(t)\frac{\text{gm}}{\text{cm}^3}\right) = rX(t)$ gm/sec. Since the amount of the drug in the organ is given by VX, where V is the constant volume of the organ, we have the following linear equation balancing inflow and outflow:

$$\frac{d}{dt}(VX) = rc - rX, \text{ or } \frac{dX}{dt} + \left(\frac{r}{V}\right)X = \frac{rc}{V}.$$

Using the integrating factor $e^{\int \frac{r}{V}dt} = e^{rt/V}$, we find that $X(t) = c + Ke^{-rt/V}$. The condition $X(0) = 0$ yields $K = -c$, so that $X(t) = c - ce^{-rt/V} = c\left(1 - e^{-rt/V}\right)$. Then $X(t) = c\left(1 - e^{-rt/V}\right) \le c_{\max} \Rightarrow 1 - \frac{c_{\max}}{c} \le e^{-rt/V} \Rightarrow \ln\left(1 - \frac{c_{\max}}{c}\right) \le -\frac{rt}{V} \Rightarrow \ln\left(\frac{c-c_{\max}}{c}\right) \le -\frac{rt}{V} \Rightarrow -\ln\left(\frac{c-c_{\max}}{c}\right) \ge \frac{rt}{V} \Rightarrow \ln 1 - \ln\left(\frac{c-c_{\max}}{c}\right) \ge \frac{rt}{V} \Rightarrow \ln\left(\frac{c}{c-c_{\max}}\right) \ge \frac{rt}{V} \Rightarrow t \le \frac{V}{r}\ln\left(\frac{c}{c-c_{\max}}\right)$.

2. Let $Q(t)$ denote the amount of salt (in pounds) in the tank at time t (in minutes). Note that the volume of fluid in the tank is not constant: At $t = 0$, the volume is 50, while at time $t > 0$ it is $50 + 5t$. The description of the inflow and the outflow yields the equation $dQ/dt = (4) \cdot \frac{1}{2} - 3Q/(50 + 5t)$. We note that $Q(0) = (50) \cdot \frac{1}{3} = 50/3$ pounds.

Overflow occurs when $t = 10$, so *we want the value of* Q *when* $t = 10$. The differential equation is linear, so we solve in the usual way: $\mu(t) = (50+5t)^{3/5}$, $\frac{d}{dt}\left[(50 + 5t)^{3/5}Q(t)\right] = 2(50 + 5t)^{3/5}$, $Q(t) = \frac{1}{4}(50 + 5t) + C(50 + 5t)^{-3/5}$. Using the initial condition, we find that $C = (50)^{8/5}/12$. Then $Q(t) = \frac{1}{4}(50 + 5t) + \frac{(50)^{8/5}}{12}(50 + 5t)^{-3/5}$ and $Q(10) = \frac{1}{4}(50 + 50) + \frac{(50)^{8/5}}{12}(50 + 50)^{-3/5} = 25 + \frac{25}{6(2)^{3/5}} \approx 27.749$ lbs. Thus the *concentration* at overflow is $27.749/100 = 0.2775$ lb/gal.

2.4 SLOPE FIELDS

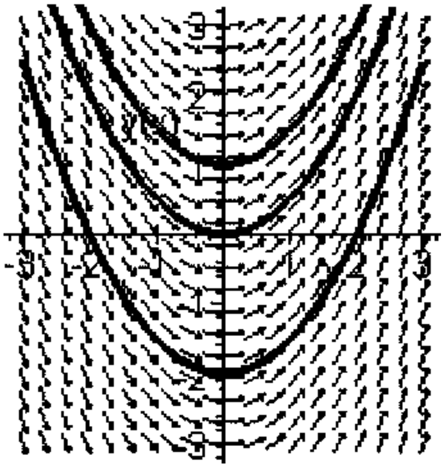
Slope fields, or direction fields, are fundamental to a geometrical (qualitative) interpretation of first-order differential equations. Students should be encouraged to get some graph paper and construct some slope fields by hand, using isoclines as suggested in the text. Graphing calculators can be programmed to produce slope fields. In *Maple*, the often-used command **Deplot** will give a slope field and, if initial conditions are provided, will produce solution curves. The corresponding instruction **PlotVectorField** does the trick in *Mathematica*.

Autonomous equations represent physical systems whose rules of evolution do not change with time. A nonautonomous system is also called a *time-dependent* system. Autonomous equations are relatively easy to analyze qualitatively because the solution curves suggested by the lineal elements do not intersect one another. Put another way, autonomous equations are characterized by their invariance with respect to all translations $t' = t + c$ of the time axis. This is the point of Exercise C1.

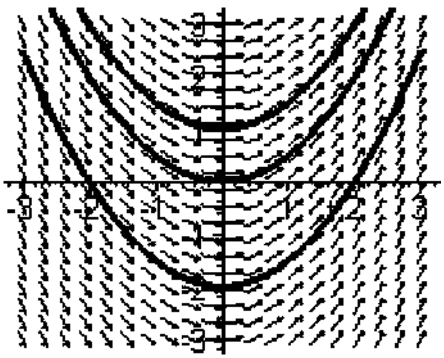
Example 2.4.3 and Exercise B7 are inspired by material produced by the Boston University Differential Equations Project. The *logistic equation* appears in Exercise B8(b). Exercise B9 comes from the classic text by Agnew, where the equation is described as having arisen in a physics problem.

A

1.



2. Note that the equations in Exercises 1 and 2 are the same.



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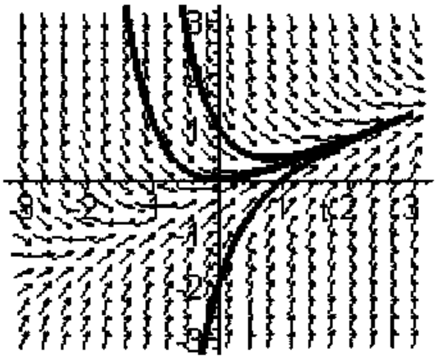

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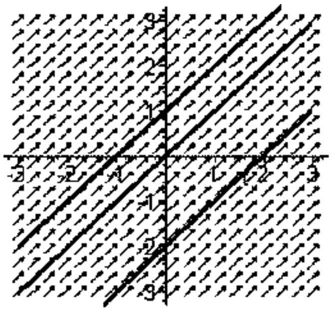

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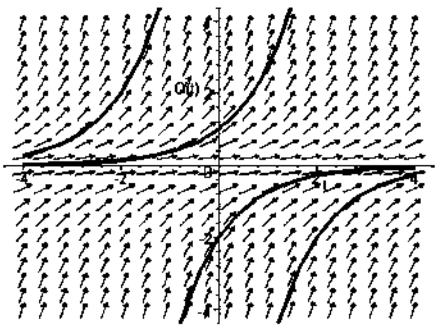

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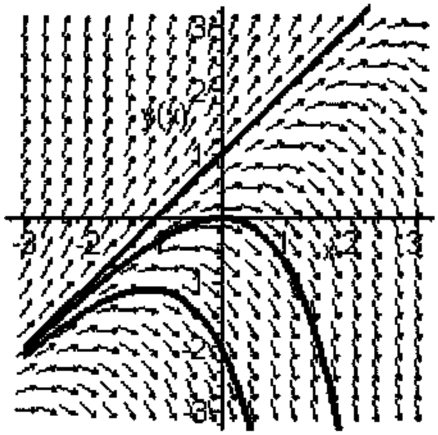

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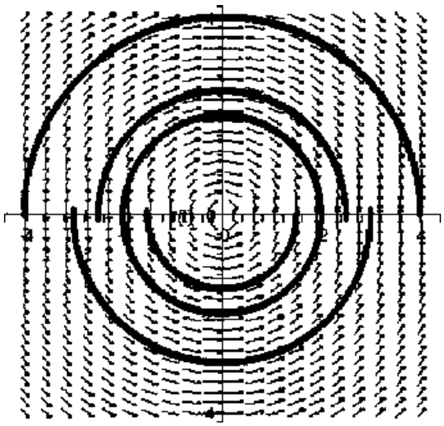

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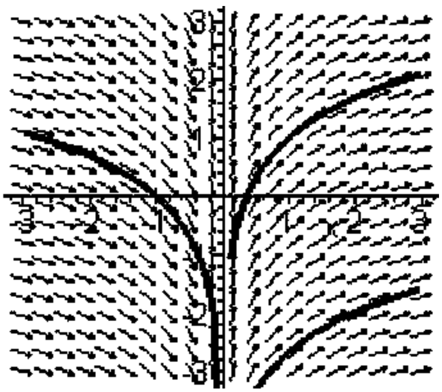

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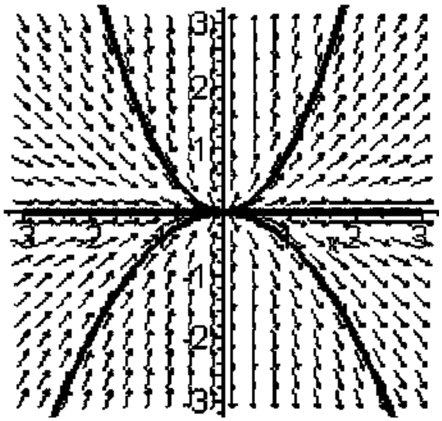

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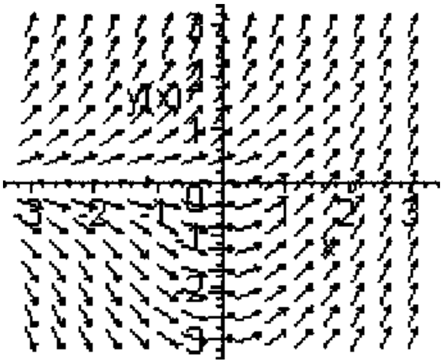

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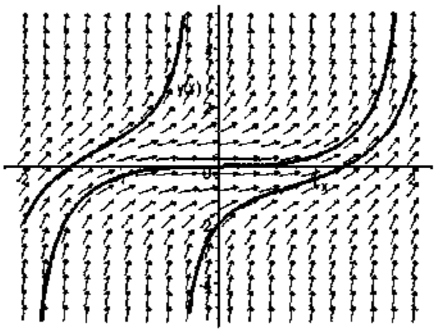

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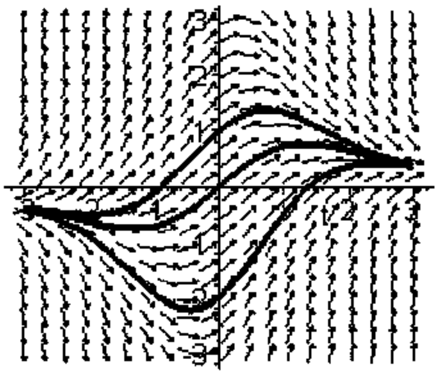

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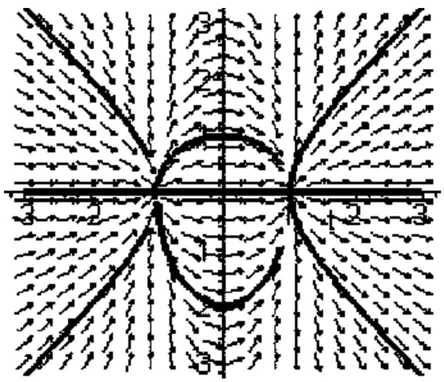
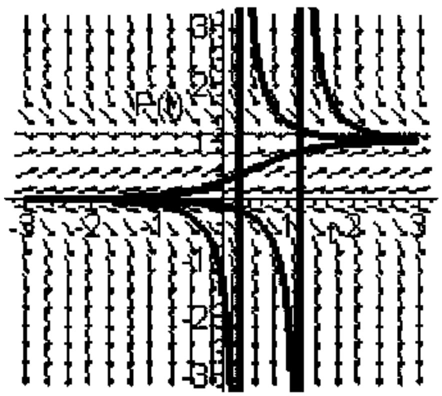


14. Some solution curves have vertical asymptotes.



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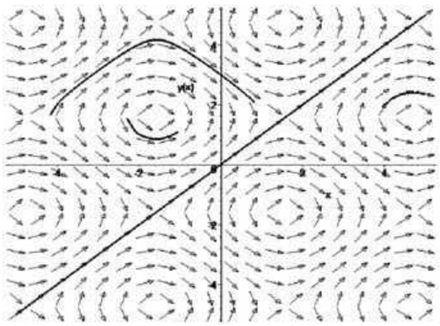
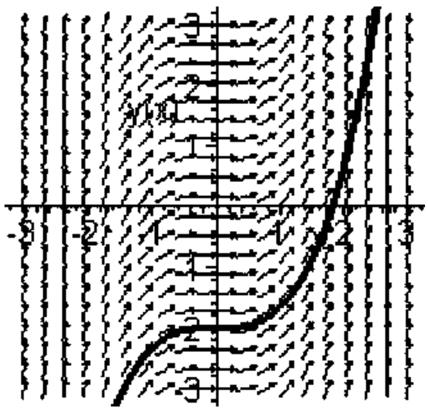
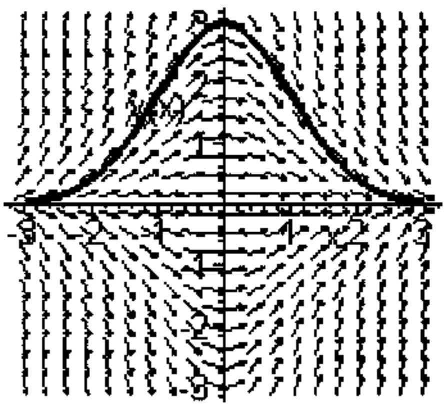


16. a.



b.



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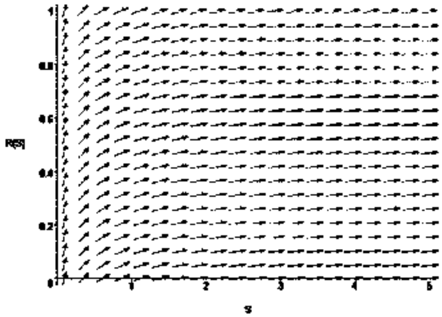
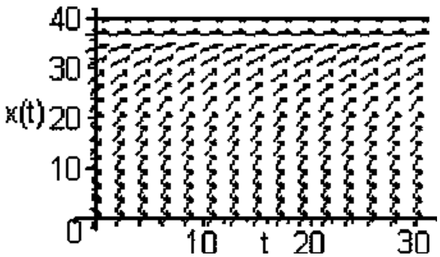


18. The isoclines are the curves defined by $dy/dt = C$, where C is a constant. In this case we have $(y + t)/(y - t) = C, y + t = C(y - t) = Cy - Ct$, so that $y - Cy = -t - Ct$, or $y = -((1 + C)/(1 - C))t$. For $C \neq 1$, this describes a one-parameter family of straight lines through the origin. For $C = 1$, we get the y-axis as the isocline.

19. The equations in Exercises 4, 5, 8, and 14 are autonomous: In each case the independent variable does not appear explicitly.

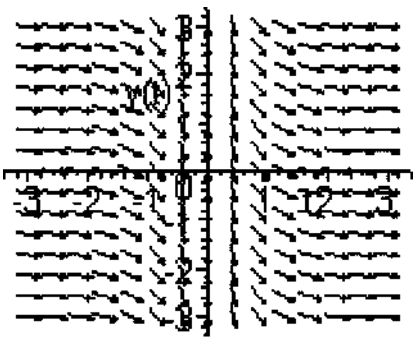
B

1. a.

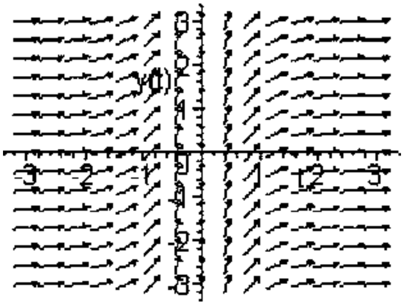


b. $x \to 40$ as $t \to \infty$.

2. If $c > 0$, we have something like



For $c < 0$, the slope field looks like



3. $\frac{1}{\sqrt{1+t^2+y^2}} = C \Rightarrow \frac{1}{1+t^2+y^2} = C^2 \Rightarrow 1 + t^2 + y^2 = \left(\frac{1}{C}\right)^2 \Rightarrow t^2 + y^2 = \left(\frac{1}{C}\right)^2 - 1$. Thus the isoclines are circles of radius $\sqrt{\left(\frac{1}{C}\right)^2 - 1}$, where $0 < |C| \leq 1$.

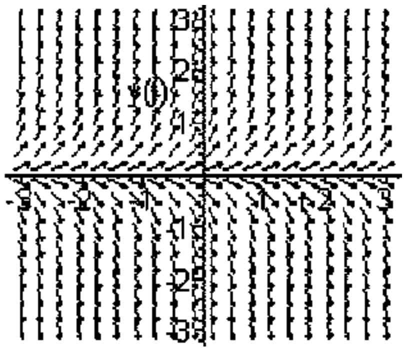
4. $xy\frac{dy}{dx} = y^2 - x^2 \Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{xy}$, and $\frac{y^2-x^2}{xy} = 0 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$. Thus the nullclines are straight lines through the origin with slopes ± 1. Actually, the origin must be omitted because of the potential division by zero.

5. In Exercise B1, we have $\frac{dx}{dt} = k(\alpha - x)(\beta - x)$, so $\frac{dx}{dt} = 0$ if and only if $x = \alpha$ or $x = \beta$.

6. Equation (a) is *nonautonomous*. Therefore along any horizontal line $y = k$ in the slope field, the slopes will change as the value of t changes. Equation (b) is also *nonautonomous*, but the slope field depends only on the value of the independent variable t. Here every solution curve has the form $y = \int_{t_0}^{t} f(x)dx + y_0$. Equation (c) is *autonomous*, so that along any horizontal line $y = k$ in the slope field, the slopes will be constant.

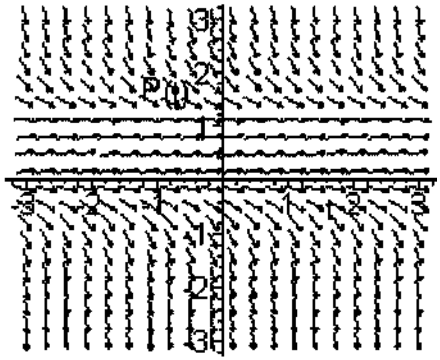
7. Equation (1) is autonomous and can only match slope field (C) or (D). Noting that $dy/dt = 0$ only for $y = -1$, we conclude that (1) matches (C). We can also see that $dy/dt < 0$ for $y < -1$, again giving us (C) as the match. Equation (2) is nonautonomous, giving us (A) or (B) as the only possible matches. Since $dy/dt = 0$ only for $y = t$, we look for horizontal "steps" along this line through the origin. Slope field (A) has this feature. We also note that $dy/dt > 0$ for $y > t$ and $dy/dt < 0$ for $y < t$, a feature present in slope field (A). Equation (3) is nonautonomous, with the vertical line $t = -1$ as its only nullcline. Furthermore, by integrating both sides of the equation with respect to t, we find that the solutions are the parabolas $y = t^2/2 + t + C$. Only slope field (B) has the two features described.

8. a.



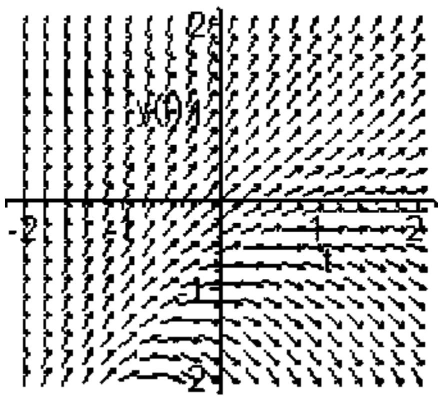
If the initial point is above the t-axis (i.e., $y(t_0) = y_0 < 0$), then $y \to \infty$ as $t \to \infty$. If the initial point is *on* the t-axis, the solution curve *is* the t-axis—that is, $y(t) = 0$ for *all* values of t. Finally, if the initial point is below the t-axis, then $y \to -\infty$ as $t \to \infty$.

b.



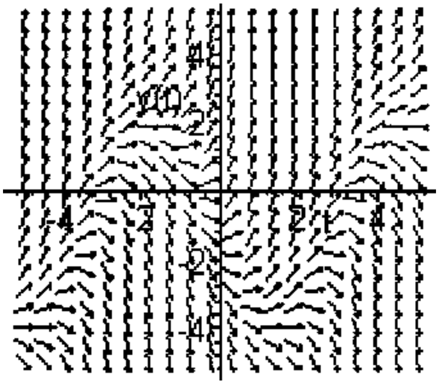
If the initial value of P, P_0, is above 1, then $P \to 1$ as $t \to \infty$. For $0 < P_0 < 1$, we have $P \to 1$ as $t \to \infty$. If $P_0 < 0$, then $P \to -\infty$ as $t \to \infty$.

c.



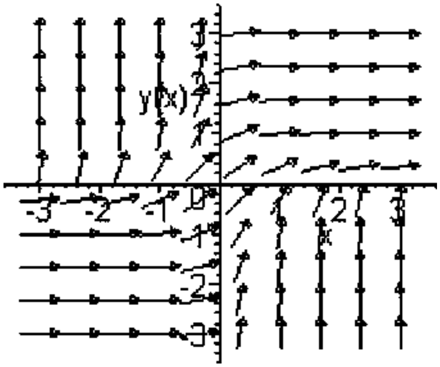
A careful examination of the slope field reveals that when $y(0) < 1/2$, we seem to have $y(t) \to -\infty$ as $t \to \infty$ and when $y(0) > 1/2$ we have $y(t) \to \infty$ as $t \to \infty$. When $y(0) = 1/2$, the solutions tend to 0.

d.



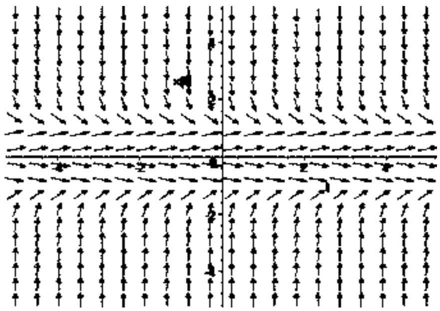
Some solutions seem to be unbounded (positively or negatively) as t tends to infinity, while others seem to be periodic. The initial condition is essential in determining which of these behaviors to expect.

9.



The slope field indicates that any solution must be an increasing function. Analytically, the fact that $\exp(-2xy)$ is always positive tells us this. Some solutions in the second quadrant seem to have vertical asymptotes, so that they "blow up in finite time," while other solutions starting out in this area flatten out (approach some finite value asymptotically) as they pass through the first quadrant. Solutions with initial points in the third quadrant are almost flat until they pass into the first or fourth quadrants. Starting out in the fourth quadrant, a solution will start out having a very large slope, but will move into the first quadrant and approach a positive finite value asymptotically. Overall then, we see that as $x \to \infty$, we have both $y \to \infty$ and $y \to a$, where a is a positive real number. As $x \to -\infty$ (i.e., as we look at the slope field from right to left), we see that $y \to \infty$ or $y \to 0$.

10. a.



b. $x(t) \to 1$ as $t \to \infty$
c. $x(t) \to 1$ as $t \to \infty$
d. $x(t) \to -1$ as $t \to \infty$
e. $x(t) \to -1$ as $t \to \infty$

C

1. Suppose that $x = \varphi(t)$ is a solution of the autonomous equation $dx/dt = f(x) = f(x(t))$. Now Consider $y(t) = \varphi(t+k)$, where k is any real number. Then $y'(t) = [\varphi(t+k)]' = \varphi'(t+k) \cdot (t+k)' = \varphi'(t+k) = f(\varphi(t+k)) = f(y(t))$, so $y(t)$ is also a solution.
2. Since $\cos t = \sin\left(t + \frac{\pi}{2}\right)$, we use Exercise C1 (with $k = \pi/2$) to conclude that $\cos t$ is also a solution of the autonomous equation.

2.5 PHASE LINES AND PHASE PORTRAITS

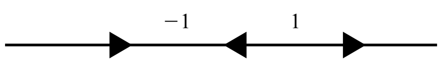
Phase lines and phase portraits are important qualitative tools for studying autonomous first-order differential equations. The extension of these ideas to autonomous *systems* of differential equations is given in Section 4.7.

The *logistic equation* is an important model that seems to have treated by anyone who has ever written about ODEs. The treatment I give is appropriate for a calculus course once the significance of the second derivative has been established.

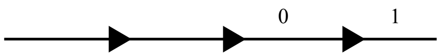
<It's useful to look at the slope fields for the equations in Exercises 1–12 as a check.>

A

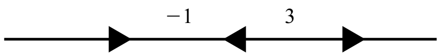
1. We have $dy/dt = y^2 - 1 = (y+1)(y-1)$, so that the critical points are $y = -1$ and $y = 1$. These points split the y-axis into three intervals: $-\infty < y < -1, -1 < y < 1$, and $1 < y < \infty$. If $y < -1$, then $dy/dt > 0$. If $-1 < y < 1$, then $dy/dt < 0$. Finally, for $y > 1$, $dy/dt > 0$. The resulting phase portrait is



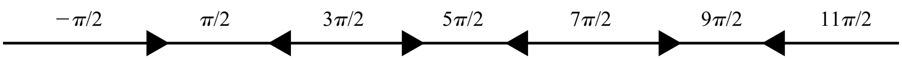
2. Here we see that $y = 0$ and $y = 1$ are the critical points. Since y' is the product of squared expressions, we have $y' > 0$ for all values of y other than 0 and 1 Thus the phase portrait looks like



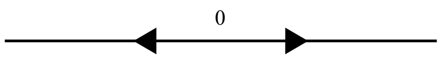
3. The critical points are $x = -1$ and $x = 3$. In the subinterval $x < -1, x' > 0$. In the subinterval $-1 < x < 3, x' < 0$. For $x > 3$, we have $x' > 0$:



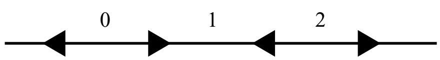
4. The function $\cos x$ is zero at odd multiples of $\pi/2$—that is, at points of the form $(2k+1)\pi/2, k = 0, \pm 1, \pm 2, \ldots$. Looking at the graph of $\cos x$, we see (for example) that $x' > 0$ on the intervals $\ldots, (-5\pi/2, -3\pi/2), (-\pi/2, \pi/2), (3\pi/2, 5\pi/2), \ldots$ and $x' < 0$ on the intervals $\ldots, (-3\pi/2, -\pi/2), (\pi/2, 3\pi/2), (5\pi/2, 7\pi/2), \ldots$. Using the fact that the cosine is an *even* function (*i.e.*, $\cos(-x) = \cos x$) so that its graph is symmetric about the y-axis, we can express our observations neatly as follows: $x' > 0$ on the intervals $(-\pi/2 + 2k\pi, \pi/2 + 2k\pi), k = 0, \pm 1, \pm 2, \ldots$ and $x' < 0$ on the intervals $(\pi/2 + 2k\pi, 3\pi/2 + 2k\pi), k = 0, \pm 1, \pm 2, \ldots$. The resulting phase portrait is



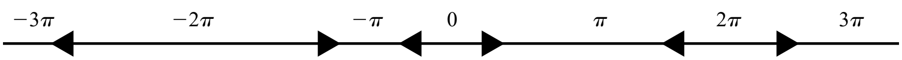
5. We have critical points where $e^y = 1$—that is, when $y = 0$. For $y < 0, y' < 0$; while for $y > 0$, we have $y' > 0$:



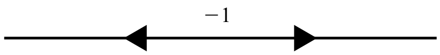
6. The critical points are $y = 0, 1,$ and 2. For $y < 0, y' < 0$. For $0 < y < 1, y' > 0$. For $1 < y < 2, y' < 0$. For $y > 2, y' > 0$. The phase portrait is



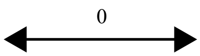
7. This is similar to Exercise 4. The critical points are $y = k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$. Examination of the graph of the sine reveals that $y' > 0$ on the intervals $(2k\pi, (2k+1)\pi), k = 0, \pm 1, \pm 2, \ldots$ and $y' < 0$ on the intervals $((2k-1)\pi, 2k\pi), k = 0, \pm 1, \pm 2, \ldots$:



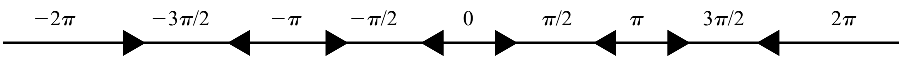
8. Technically, there is no critical point for this equation since $x' = 1 - x/(1+x) = 1/(1+x)$ is never zero. However, we can see that x' fails to exist $x = -1$, so that we can focus on this point. We see that $x' > 0$ for $x > -1$ and $x' < 0$ for $x < -1$:



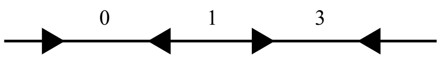
9. The only critical point is $y = 0$. It is easy to see that $y' > 0$ when $y > 0$ and $y' < 0$ when $y < 0$:



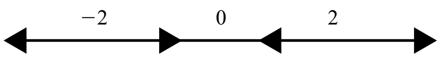
10. The critical points are those values of y for which either $\sin y = 0$ or $\cos y = 0$. This is the set of values given by $y = k\pi/2, k = 0, \pm 1, \pm 2, \ldots$. If we look at the graph of $\sin y \cos y$, we see that $\dot{y} > 0$ on the intervals $\left(k\pi, (k + \frac{1}{2})\pi\right), k = 0, \pm 1, \pm 2, \ldots$ and $\dot{y} < 0$ on the intervals $\left((k - \frac{1}{2})\pi, k\pi\right), k = 0, \pm 1, \pm 2, \ldots$:



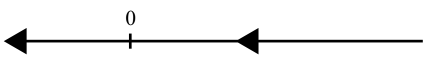
11. The critical points are $x = 0, 1$, and 3. For $x < 0, dx/dt > 0$. For $0 < x < 1, dx/dt < 0$. For $1 < x < 3$, we have $dx/dt > 0$. Finally, for $x > 3, dx/dt < 0$. The phase portrait is



12. The critical points are $y = -2, 0$, and 2. For $y < -2, dy/dt < 0$. For $-2 < y < 0$, we have $dy/dt > 0$. For $0 < y < 2, dy/dt < 0$. For $y > 2, dy/dt > 0$. The phase portrait is

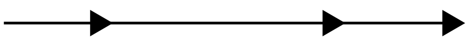


13. The only critical point is $x = 0$. If $x < 0, dx/dt < 0$. If $x > 0$, we have $dx/dt < 0$ again. The phase portrait is



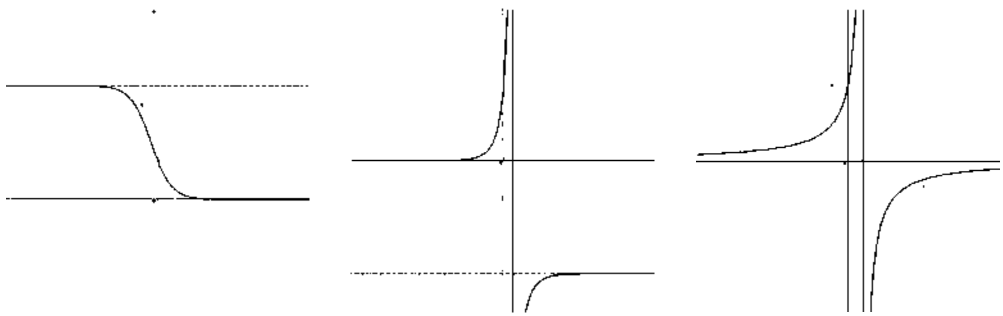
14. From the phase portrait in Exercise 11, we see that if the population $x(t)$ falls below 1 (hundred, thousand, million,...), then $x(t) \to 0$ as $t \to \infty$—that is, the creatures being modeled become extinct.

15. a. There are *no* critical points because $-\frac{1}{2} \le \frac{1}{2}\cos x \le \frac{1}{2}$ for all values of x. Thus $\dot{x} > 0$ for all values of x. The phase portrait is



b. All solutions are monotonically increasing. (You can also examine the slope field of this autonomous equation.)

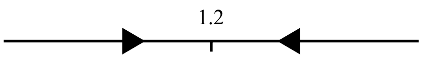
16.



$a = 0$ $a > 0$ $a < 0$

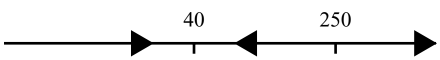
B

1. **a.** We can rewrite the equation as $\frac{dI}{dt} = 24 - 20I$. Setting the derivative equal to zero, we see that the only critical point is $I = 6/5 = 1.2$. When $I < 1.2$, we have $\frac{dI}{dt} > 0$; and when $I > 1.2$, we see that $\frac{dI}{dt} < 0$. The phase portrait is



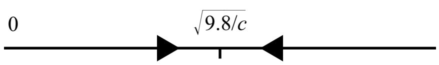
b. If the initial current, $I(0)$, is 3 amps, it is to the right of the critical point, so that the current tends to *decrease* toward 1.2 amps as t gets larger.

2. **a.** We have $\frac{dx}{dt} = k(250 - x)(40 - x)$. Setting the derivative equal to zero, we have the critical points $x = 40$ and $x = 250$. If $x < 40$, then $(250 - x) > 0$ and $(40 - x) < 0$, so that $dx/dt > 0$ (remembering that $k > 0$). If $40 < x < 250$, then $(250 - x) > 0$ and $(40 - x) < 0$, so that $dx/dt < 0$. Finally, if $x > 250$, then $(250 - x) < 0$ and $(40 - x) < 0$, so that $dx/dt > 0$. The phase portrait is



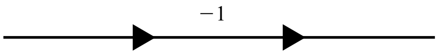
b. If $x(0) = 0$, then $dx/dt > 0$ and x increases toward 40 as $t \to \infty$. (See the slope field for problem B1 in Exercises 2.4.)

3. **a.** The critical points are $v = \pm\sqrt{9.8/c}$. The phase portrait is



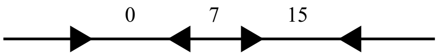
b. The positive equilibrium solution is $v = \sqrt{9.8/c}$, which is a *sink*. (This value of v is called the *terminal velocity*.)

4. We see that $y = -1$ is the only critical point. Since dy/dt is a square, $dy/dt \geq 0$ for all values of y. Thus any solution y must be a nondecreasing function. The phase portrait is



a. If $y(0) > -1$, the graph of $y(t)$ must increase *away* from $y = -1$ as t increases.
b. If $y(0) < -1$, the graph of $y(t)$ must increase *toward* the line $y = -1$ as t increases—that is, the line $y = -1$ is a horizontal asymptote for such a solution. An analysis of d^2y/dt^2 reveals that y is concave up when $y > -1$ and concave down when $y < -1$.

5. **a.** The critical points are $P = 0, 7$, and 15. For $P < 0$, $dP/dt > 0$. For $0 < P < 7$, we have $dP/dt < 0$. For $7 < P < 15$, $dP/dt > 0$. Finally, for $P > 15$, $dP/dt < 0$. The phase portrait is



Since the initial condition falls in the interval $0 < P < 7$, we see that $P(t)$ is decreasing toward the t-axis—that is, $P(t) \to 0$ as $t \to \infty$.

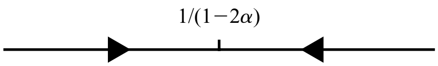
6. **a.** For example, $x' = x(x-1)^2$.
b. For example, $x' = -(x+5)(x+5) = 25 - x^2$.
c. For example, $x' = (x-2)^2(x-7)$.

7. See the graph for B7 on p. 453 of the text.

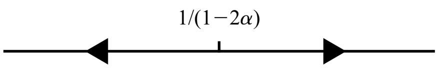
8. One possible answer is $dQ/dt = Q(Q+1)^2(Q-2)$. Exercises A3 and A6 provided some clues.

C

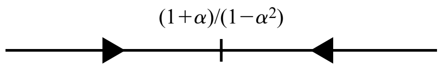
1. The critical point is $x = -1/(2\alpha - 1) = 1/(1 - 2\alpha), x \neq 1/2$.
Case 1: $\boxed{\alpha < 1/2}$ This implies that $2\alpha - 1 < 0$ and so $1/(2\alpha - 1) < 0$. Consequently, if $x < 1/(1-2\alpha) = -1/(2\alpha - 1)$, then $(2\alpha - 1)x > -1$ and $(2\alpha - 1)x + 1 > 0$. If $x > 1/(1-2\alpha) = -1/(2\alpha - 1)$, then $(2\alpha - 1)x < -1$ and $(2\alpha - 1)x + 1 < 0$. The phase portrait shows that $x(t) \to 1/(1-2\alpha)$ as $t \to \infty$.



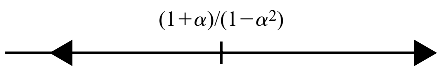
Case 2: $\boxed{\alpha > 1/2}$ This implies that $2\alpha - 1 > 0$ and so $1/(2\alpha - 1) > 0$. Consequently, if $x < 1/(1-2\alpha) = -1/(2\alpha - 1)$, then $(2\alpha - 1)x < -1$ and $(2\alpha - 1)x + 1 < 0$. If $x > 1/(1-2\alpha) = -1/(2\alpha - 1)$, then $(2\alpha - 1)x > -1$ and $(2\alpha - 1)x + 1 > 0$. The phase portrait shows that $x(t) \to +\infty$ as $t \to \infty$.



2. $x' = (\alpha^2 - 1)x + 1 + \alpha : x' = 0$ implies that $x = (1+\alpha)/(1-\alpha^2)$.
Case 1: $\boxed{-1 < \alpha < 1}$ This implies that $\alpha^2 - 1 < 0$. Then $x < (1+\alpha)/(1-\alpha^2)$ implies $(\alpha^2 - 1)x > (\alpha^2 - 1)(1+\alpha)/(1-\alpha^2) = -(1+\alpha)$, so $(\alpha^2 - 1)x + 1 + \alpha > 0$. Similarly, if $x > (1+\alpha)/(1-\alpha^2)$, then $(\alpha^2 - 1)x < (\alpha^2 - 1)(1+\alpha)/(1-\alpha^2) = -(1+\alpha)$, so $(\alpha^2 - 1)x + 1 + \alpha < 0$. The phase portrait shows that $x(t) \to (1+\alpha)/(1-\alpha^2)$ as $t \to \infty$.



Case 2: $\boxed{|\alpha| > 1}$ This is equivalent to $\alpha < -1$ or $\alpha > 1$. Thus $\alpha^2 - 1 > 0$ and $(1+\alpha)/(1-\alpha^2) < 0$. Now if $x < (1+\alpha)/(1-\alpha^2)$, then $(\alpha^2-1)x < (\alpha^2-1)(1+\alpha)/(1-\alpha^2) = -(1+\alpha)$, so $(\alpha^2 - 1)x + 1 + \alpha < 0$. On the other hand, if $x > (1+\alpha)/(1-\alpha^2)$, then we see that $(\alpha^2 - 1)x + 1 + \alpha > 0$. The phase portrait shows that $x(t) \to +\infty$ as $t \to \infty$.



Note that if $\alpha = -1$, then $x' = 0$, so x is a constant function.

3. a. $\frac{dx}{dt} = ax - bx^3 = x(a - bx^2) = 0 \Leftrightarrow x = 0, -\sqrt{a/b}$, or $\sqrt{a/b}$. We have four intervals to examine: $\left(-\infty, -\sqrt{a/b}\right)$, $\left(-\sqrt{a/b}, 0\right)$, $\left(0, \sqrt{a/b}\right)$, and $\left(\sqrt{a/b}, \infty\right)$.

(1) If $x < -\sqrt{a/b}$, then $x < 0$ and $x^2 > a/b, bx^2 > a, -bx^2 < -a, a - bx^2 < 0$, so that $dx/dt > 0$.

(2) If $-\sqrt{a/b} < x < 0$, then $x < 0$ and $x^2 < a/b, bx^2 < a, -bx^2 > -a, a - bx^2 > 0$, so that $dx/dt < 0$.

(3) If $0 < x < \sqrt{a/b}$, then $x > 0$ and $x^2 < a/b, bx^2 < a, -bx^2 > -a, a - bx^2 > 0$, so that $dx/dt > 0$.

(4) When $x > \sqrt{a/b}$, then $x > 0$ and $x^2 > a/b, bx^2 > a, -bx^2 < -a, a - bx^2 < 0$, so that $dx/dt < 0$.

Using the preceding analysis, we can draw the following phase portrait: [See phase portrait for C3a. on p. 454 of the text.]

b. If $x(0)$ is slightly larger than $\sqrt{a/b}$, the phase portrait indicates that $x(t)$ will decrease to $\sqrt{a/b}$ as t increases.

c. If $x(0) = 0$, then $x(t)$ stays at zero as t increases. Since $dx/dt = 0$ when $x = 0$, we see that there is no change in the value of $x(t)$ at this point.

d. If $x(0)$ is slightly smaller than $\sqrt{a/b}$, the phase portrait indicates that $x(t)$ will increase to $\sqrt{a/b}$ as t increases.

2.6 EQUILIBRIUM POINTS: SINKS, SOURCES, AND NODES

This section continues to develop the qualitative analysis begun in the previous section. This material falls under the heading of *stability theory*, and is restricted to the study of the asymptotic behavior of solutions of autonomous first-order ODEs. These concepts will be extended to systems in Chapter 4, and the behavior of solutions of autonomous linear systems will be analyzed completely in Chapter 5. Chapter 7 applies these ideas to nonlinear systems.

Given an autonomous equation $x' = f(x)$, an equilibrium solution ψ is said to be an *asymptotically stable solution* (*sink, attractor*) if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution φ of the equation satisfying $|\varphi(0) - \psi(0)| < \delta$ satisfies $|\varphi(t) - \psi(t)| < \varepsilon$ for $t \geq 0$ and $|\varphi(t) - \psi(t)| \to 0$ as $t \to \infty$. This idea is also referred to as *Lyapunov stability*. Similar precise definitions can be given to sources and nodes.

Exercise A16 asks the student to consider a simple version of a *balance equation*. Balance equations, also called *conservation equations*, are mathematical statements of the fact that what goes into a system must be stored, come out, or be transformed into something else: matter, energy, and momentum can't just appear or disappear without explanation. Equation (2.3.1), a fundamental principle in working with compartment models, is an equation of this kind. Students who are studying ecology or environmental engineering will see these types of equations many times.

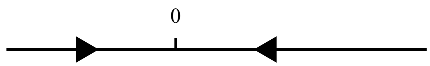
For more information on the *Gompertz equation* given in Exercise B3, see *Mathematical Models in Biology* by L. Edelstein-Keshet (McGraw-Hill, 1988; reissued by SIAM) or "Models for Growth" by E. B. Appelbaum (*College Math. J* **32** (2001), 258–259), an article written by someone who has suffered from cancer. Exercise C1 asks the student to give a qualitative analysis of the logistic equation with *harvesting* (or *culling*).

< Confirm the answers for Exercises 1–12 by looking at slope fields or phase portraits.>

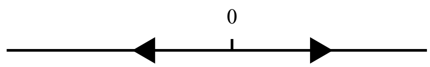
A

1. $f(y) = y^2(1-y)^2$ and $f'(y) = 2y(2y-1)(y-1)$. The equilibrium points are $y = 0$ and $y = 1$. Now $f'(0) = 0$ and $f'(1) = 0$, so that the Derivative Test fails. However, we see that $f'(y) < 0$ for values of y below 0 and $f'(y) > 0$ for values of y above 0, so that $y = 0$ behaves like both a sink and a source and so is a **node**. Similarly, $f'(y) < 0$ for values of y just below 1 and $f'(y) > 0$ for values above 1, indicating that $y = 1$ is also a **node**.
2. $f(x) = \cos x$ and $f'(x) = -\sin x$. The equilibrium points are the odd multiples of $\pi/2$—that is, points of the form $(2k+1)\pi/2, k = 0, \pm 1, \pm 2, \ldots$. Now $f'((2k+1)\pi/2) = -\sin((2k+1)\pi/2) = -\sin(k\pi + \pi/2) = -\cos(k\pi) = (-1)^{k+1}$. (If you don't remember these trigonometric facts, just look at the graph of $y = -\sin x$ to see the pattern.) Thus $f'(x) > 0$ and x is a **source** when k is *odd*—that is, when $x = \ldots -9\pi/2, -5\pi/2, -\pi/2, 3\pi/2, 7\pi/2, \ldots$—and $f'(x) < 0$, so that x is a **sink**, when $x = \ldots -7\pi/2, -3\pi/2, \pi/2, 5\pi/2, 9\pi/2, \ldots$.

3. $f(y) = e^y - 1$ and $f'(y) = e^y$. The only equilibrium point is $y = 0$. Since $f'(0) = 1 > 0$, we see that $y = 0$ is a **source**.
4. $f(y) = y^2(y^2 - 1)$ and $f'(y) = 2y(2y^2 - 1)$. The equilibrium points are $y = -1, 0$, and 1. Now $f'(-1) = -2 < 0$, so that $y = -1$ is a **sink**; and $f'(1) = 2 > 0$, so that $y = 1$ is a **source**. We have $f'(0) = 0$, so that our usual test fails. Since $f'(y) < 0$ for values of y just less than 0 and $f'(y) > 0$ for values of y just greater than 0, we conclude that $y = 0$ is a **node**.
5. $f(x) = ax + bx^2$ and $f'(x) = a + 2bx$. The equilibrium points are $x = -a/b$ and 0. Since $f'(-a/b) = -a < 0$, we see that $x = -a/b$ is a **sink**. Since $f'(0) = a > 0$, we conclude that $x = 0$ is a **source**.
6. $f(x) = x^3 - 1$ and $f'(x) = 3x^2$. The only equilibrium point is $x = 1$. Since $f'(1) = 3 > 0$, we see that $x = 1$ is a **source**.
7. $f(x) = x^2 - x^3$ and $f'(x) = x(2 - 3x)$. The equilibrium points are $x = 0$ and 1. Since $f'(1) = -1 < 0, x = 1$ is a **sink**. But $x = 0$ is a **node** because $f'(x) > 0$ for values of x just below and just above 0.
8. $f(y) = 10 + 3y - y^2 = (2 + y)(5 - y)$ and $f'(y) = 3 - 2y$. The equilibrium points are $y = -2$ and 5. Since $f'(-2) = 7 > 0, y = -2$ is a **source**. Since $f'(5) = -7 < 0, y = 5$ is a **sink**.
9. $f(x) = x(2 - x)(4 - x)$ and $f'(x) = 3x^2 - 12x + 8$. The equilibrium points are $x = 0, 2$, and 4. Since $f'(0) = 8 > 0, x = 0$ is a **source**. Since $f'(2) = -4 < 0, x = 2$ is a **sink**. Since $f'(4) = 8 > 0, x = 4$ is a **source**.
10. $f(x) = -x^3$ and $f'(x) = -3x^2$. The only equilibrium point is $x = 0$. Since $f'(0) = 0$, we investigate further. Since $f'(x) < 0$ for every nonzero value of x, we conclude that $x = 0$ is a **sink**. The phase portrait confirms this:

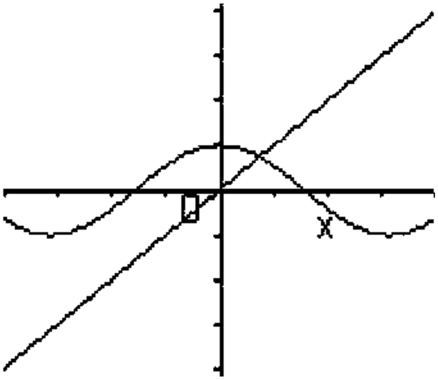


11. $f(x) = x^3$ and $f'(x) = 3x^2$. The only equilibrium point is $x = 0$. Since $f'(0) = 0$, we examine the situation more carefully. Since $f'(x) > 0$ for every nonzero value of x, we conclude that $x = 0$ is a **source**. The phase portrait confirms this:



12. $f(y) = y\ln(y+2)$ and $f'(y) = y/(y+2) + \ln(y+2)$. The equilibrium points are $y = -1, 0$. Since $f'(-1) = -1 < 0$, we know that $y = -1$ is a **sink**. Since $f'(0) = \ln 2 > 0, y = 0$ is a **source**.
13. $f(x) = x - \cos x$ and $f'(x) = 1 + \sin x$. Any equilibrium point x^* must satisfy the equation $x^* - \cos x^* = 0$, or $x^* = \cos x^*$. We graph $y = x$ and $y = \cos x$ separately on the same set

axes and notice that the graphs intersect at exactly one point:



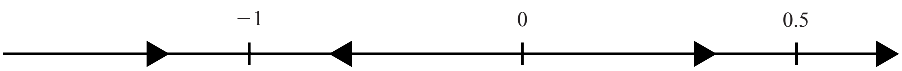
This point of intersection x^* is the only equilibrium point. Since the graph indicates that x^* lies between 0 and $\pi/2$, we see that $f'(x^*) = 1 + \sin x^* > 1 > 0$, so that this equilibrium point (for which we have no explicit formula) is a **source**.

14. $f(x) = x - e^{-x}$ and $f'(x) = 1 + e^{-x}$. The only equilibrium point is $x \approx 0.5671$. Since $f'(x) > 0$ for all x, $f'(0.5671) > 0$ and so $x = 0.5671$ is a **source**.

15. $x' = x(x+1)(x-0.5)^6$: The equilibrium points are $x = -1, 0$, and 0.5. It will be easier to examine the signs of x' than to use the Derivative Test.

x	Sign of x	Sign of $x+1$	Sign of $(x-0.5)^6$	Sign of x'
$-\infty < x < -1$	$-$	$-$	$+$	$+$
$-1 < x < 0$	$-$	$+$	$+$	$-$
$0 < x < 0.5$	$+$	$+$	$+$	$+$
$x > 0.5$	$+$	$+$	$+$	$+$

The phase portrait that can be drawn using the information from the table shows that $x = -1$ is a **sink**, $x = 0$ is a **source**, and $x = 0.5$ is a **node**.



16. **a.** This problem describes a *compartment* model (see Section 2.3).
$\frac{dQ}{dt} = D(Q^* - Q) = 0 \Leftrightarrow Q = Q^*$.

b. When $Q < Q^*$, $dQ/dt > 0$ and when $Q > Q^*$, $dQ/dt < 0$. Therefore the equilibrium solution is a *sink* and therefore **stable**. You could also use the Derivative

Test with $f(Q) = D(Q^* - Q)$ and $f'(Q) = -D : f'(Q^*) = -D < 0$ since D is positive.

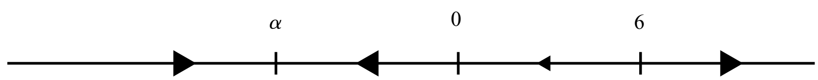
B

1. a. $M\frac{du}{dt} = \frac{8P}{u} - bSu^2 = 0 \Rightarrow 8P - bSu^3 = 0 \Rightarrow u = \sqrt[3]{\frac{8P}{bS}} = 2\sqrt[3]{\frac{P}{bS}}$.

   b. Since M represents mass, a positive constant in this case, it has no effect on the sign of du/dt. Therefore we take $f(u) = \frac{8P}{u} - bSu^2$, so that $f'(u) = -\frac{8P}{u^2} - 2bSu$ and $f'\left(2\sqrt[3]{\frac{P}{bS}}\right) = \frac{-8P}{4\left(\frac{P}{bS}\right)^{\frac{2}{3}}} - 2bS\left(2\sqrt[3]{\frac{P}{bS}}\right) = -6P^{\frac{1}{3}}(bS)^{\frac{2}{3}} < 0$, so that $u = 2\sqrt[3]{\frac{P}{bS}}$ is a **sink**.

   c. The fact that the equilibrium speed $2\sqrt[3]{\frac{P}{bS}}$ is a sink suggests that a rower may start from rest with maximum acceleration but then tire a bit so that his or her speed would level off at the equilibrium speed. If we observe the rower at a time when his or her speed is *greater* than the equilibrium speed, then we can reasonably believe that he or she may tire or the "drag force" bSu^2 may exceed the "tractive force" $\frac{8P}{u}$ and so slow the boat down.

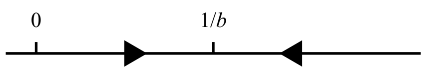
2. a.



   b. From part (a) we see that a is a **sink** and b is a **source**. We can also see this by noticing that $f'(a) < 0$ and $f'(b) > 0$.

3. $f(N) = -aN\ln(bN)$ and $f'(N) = -a(1 + \ln(bN))$. Since the nature of the model implies that $N > 0$ and since $-aN < 0$, the sign of $f(N)$ depends on the sign of $\ln(bN)$. For $0 < bN < 1$, $\ln(bN) < 0$. Thus $f(N) = -aN\ln(bN) > 0$ for $0 < N < 1/b$. For $bN > 1$ (that is, for $N > 1/b$), $\ln(bN) > 0$, so that $f(N) = -aN\ln(bN) < 0$.

   a. The phase portrait for this equation is

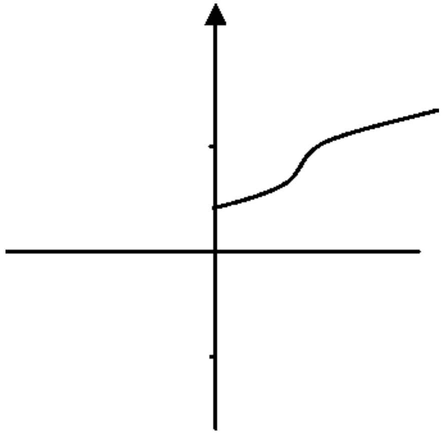


   b. A possible graph of $f(N)$ against N (with $a = 10$ and $b = 2$) is on p. 454 of the text.

   c. $f(N)$ is undefined at $N = 0$, so that the only equilibrium point is $N = 1/b$. The phase portrait given in (a) indicates that this is a **sink**. Also, $f'(1/b) = -a(1 + \ln(1)) = -a < 0$.

   d. Assume that $0 < N \le 1$ We have $\ddot{N}(t) = f'(N) = -a(1+\ln(bN)) > 0$, so that the graph of $N(t)$ is concave up, when $1+\ln(bN) < 0$—that is, when $0 < N < 1/be < 1/b$ (where e is the base of the natural log). Similarly, $\ddot{N}(t) < 0$ and $N(t)$ is concave down when $N > 1/be$.

e. Putting together the information from (c) and (d), we can sketch $N(t)$:



4. There is no such equation. Between any two sinks, there must be a source. Suppose there were three sinks, x_1, x_2, and x_3, with $x_1 < x_2 < x_3$. If we consider the phase portrait with just x_1 and x_3 plotted, notice that any point between these two must behave like a *source*:

C

1. **a.** $\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) - h = 0 \Rightarrow rP(k - P) - hk = 0 \Rightarrow -rP^2 + rkP - hk = 0 \Rightarrow$ [by the Quadratic Formula] $P = \frac{-rk \pm \sqrt{(rk)^2 - 4(-r)(-hk)}}{-2r} = \frac{-rk \pm \sqrt{rk(rk-4h)}}{-2r} = \frac{rk \mp \sqrt{rk(rk-4h)}}{2r}$. If $rk(rk - 4h) > 0$, there are distinct real (and non-zero) solutions. Since $rk > 0$, we have these two solutions if $rk - 4h > 0$, or $h < \frac{rk}{4}$.

b. The smaller of the equilibrium solutions in (a) is $P_S = \frac{rk - \sqrt{rk(rk-4h)}}{2r}$. Let $f(P) = rP\left(1 - \frac{P}{k}\right) - h = rP - \left(\frac{r}{k}\right)P^2$, so that $f'(P) = r - \left(\frac{2r}{k}\right)P$ and $f'(P_S) = r - \left(\frac{2r}{k}\right)\left(\frac{rk - \sqrt{rk(rk-4h}}{2r}\right) = r - \left(r - \sqrt{rk(rk-4h)}\right) = \sqrt{rk(rk-4h} > 0$. By the Derivative Test, P_S is a **source**. Similarly, the larger of the equilibrium solutions is $P_L = \frac{rk + \sqrt{rk(rk-4h)}}{2r}$ and $f'(P_L) = r - \left(\frac{2r}{k}\right)\left(\frac{rk + \sqrt{rk(rk-4h}}{2r}\right) = -\sqrt{rk(rk-4h} < 0$, so that P_L is a **sink.**

2. $f(x) = -x^3 + (1 + \alpha)x^2 - \alpha x = -x(x - \alpha)(x - 1)$ and $f'(x) = -3x^2 + 2(1 + \alpha)x - \alpha$. Thus the equilibrium solutions are $x = 0, \alpha$, and 1.

a. $\boxed{\alpha < 0}$ $f'(0) = -\alpha > 0$, so $x = 0$ is a **source**. Also, $f'(\alpha) = \alpha(1 - \alpha) < 0$, so $x = \alpha$ is a **sink**; and $f'(1) = \alpha - 1 < 0$, so $x = 1$ is a **sink**.

b. $\boxed{0 < \alpha < 1}$ $f'(0) = -\alpha < 0$, so $x = 0$ is a **sink**. Then $f'(\alpha) = \alpha(1 - \alpha) > 0$, so $x = \alpha$ is a **source**. Finally, $f'(1) = \alpha - 1 < 0$, so $x = 1$ is a **sink**.

c. $\boxed{\alpha > 1}$ $f'(0) = -\alpha < 0$, so $x = 0$ is a **sink**. Also, $f'(\alpha) = \alpha(1-\alpha) < 0$, so $x = \alpha$ is a **sink**. Finally, $f'(1) = \alpha - 1 > 0$, so $x = 1$ is a **source**.

d. $\boxed{\alpha = 0}$ Then $f(x) = -x^3 + x^2 = x^2(1-x)$ and $f'(x) = -3x^2 + 2x$. The equilibrium solutions are $x = 0$ and $x = 1$. We see that $f'(0) = 0$, so the Derivative Test is inconclusive. However, for $x < 0, \dot{x} = f(x) > 0$; for $0 < x < 1, \dot{x} > 0$; and for $x > 1, \dot{x} < 0$. This tells us that $x = 0$ is a **node**, whereas $x = 1$ is a **sink**.

e. $\boxed{\alpha = 1}$ Then $f(x) = -x^3 + 2x^2 - x = -x(x-1)^2$ and $f'(x) = -3x^2 + 4x - 1$. The equilibrium solutions are $x = 0$ and $x = 1$. Now $f'(0) = -1 < 0$, so $x = 0$ is a **sink;** and $f'(1) = 0$, so the Derivative Test fails. However, $\dot{x} < 0$ for $0 < x < 1$ and $\dot{x} < 0$ for $x > 1$, so $x = 1$ is a **node**.

*2.7 Bifurcations

This section is optional. The concept of a bifurcation appears here and there in later exercises, but the explanations are self-contained. Strogatz's book, *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry and Engineering* (Addison-Wesley, 1994), contains good examples.

Instead of analyzing the behavior of the quadratic function $x^2 + x + c$ via the completion of squares as in equation (2.7.1), I sometimes use the quadratic formula immediately.

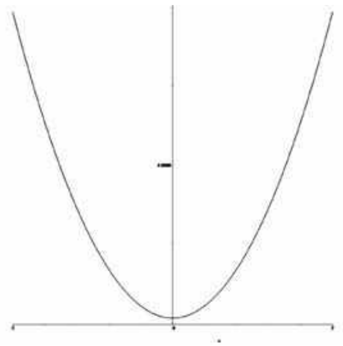
My experience has been that the concept of a *bifurcation diagram* needs slow, careful explanation. It sometimes takes a while for students to digest the fact that they are graphing the solution of an autonomous equation against the values of a parameter.

The treatment of the laser in Example 2.7.3 is adapted from Strogatz's book.

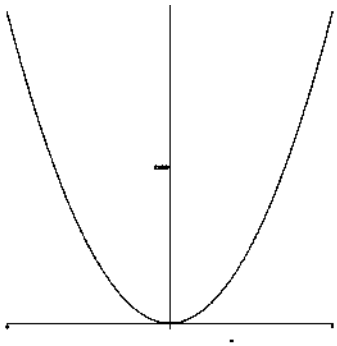
Exercise B1 is a special case of Exercise C1 of the last section, which is now revealed as a bifurcation problem. For some background on the biology and mathematics of Exercise C2, see Section 7.5 of *Mathematical Models in Biology* by L. Edelstein-Keshet (McGraw-Hill, 1988; reissued by SIAM).

A

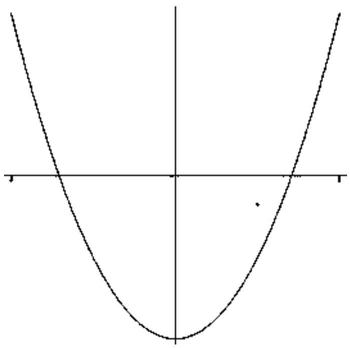
1. (1) (Also, see answer on p. 455 of the text.)



$c < 0$



$c = 0$

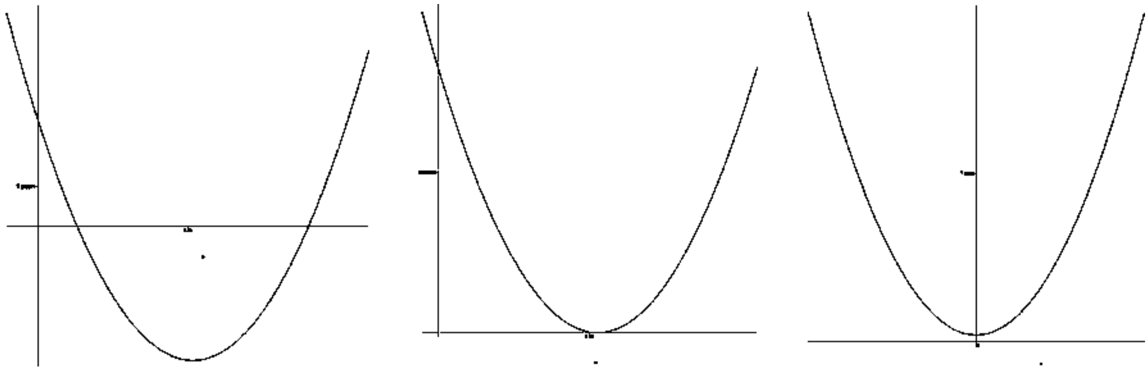


$c > 0$

(2) Clearly the number and the nature of the equilibrium solution(s) depend on the value of c. If $c = 0$, there is only one equilibrium solution, $x = 0$. If $c < 0$, there is no equilibrium solution; and if $c > 0$, there are two equilibrium solutions, $x = \pm\sqrt{c}$. The only bifurcation point is $c = 0$.

(3) See bifurcation diagram on p. 455 of the text.

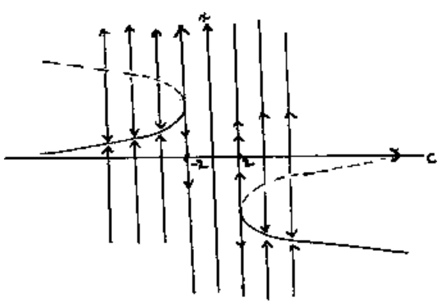
2. (1)



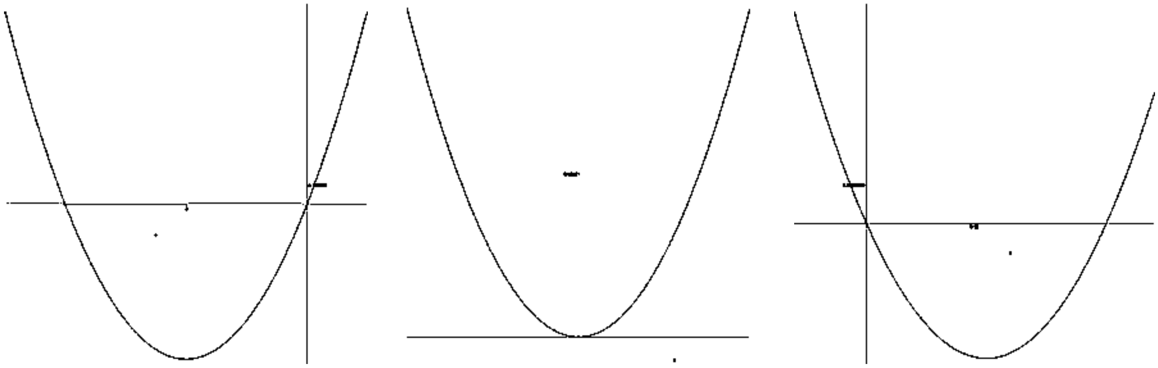
(2) We have $f(x) = 1 + cx + x^2 = 0 \Leftrightarrow x = \frac{-c\pm\sqrt{c^2-4}}{2}$. Clearly the number and the nature of the equilibrium solution(s) depend on the value of the discriminant $c^2 - 4$. Note that if $c = -2$ or $c = 2$, there is only one equilibrium point, $x = -c/2$. If $c < -2$, then $c^2 - 4 > 0$ and there are two real equilibrium points given by the formula above. When $-2 < c < 2$, we have $c^2 - 4 < 0$, so that there is *no* equilibrium solution. Finally, for $c > 2, c^2 - 4 > 0$ and there are two equilibrium solutions again. The bifurcation points are $c = -2$ and $c = 2$.

We note that $f'(x) = c + 2x$ and apply the Derivative Test to determine the nature of the equilibrium solutions: When $c < -2$ or $c > 2$, $f'\left(\frac{-c\pm\sqrt{c^2-4}}{2}\right) = c + 2\left(\frac{-c\pm\sqrt{c^2-4}}{2}\right) = \pm\sqrt{c^2 - 4}$, so that the equilibrium solution with the positive square root is a source and the equilibrium solution with the negative square root is a sink. If $c = -2$ or 2, you should check to see that you have a *node*. (The Derivative Test fails to show this.)

(3)



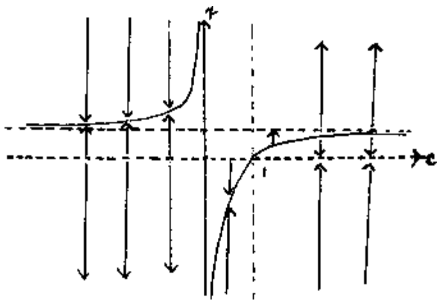
3. (1)



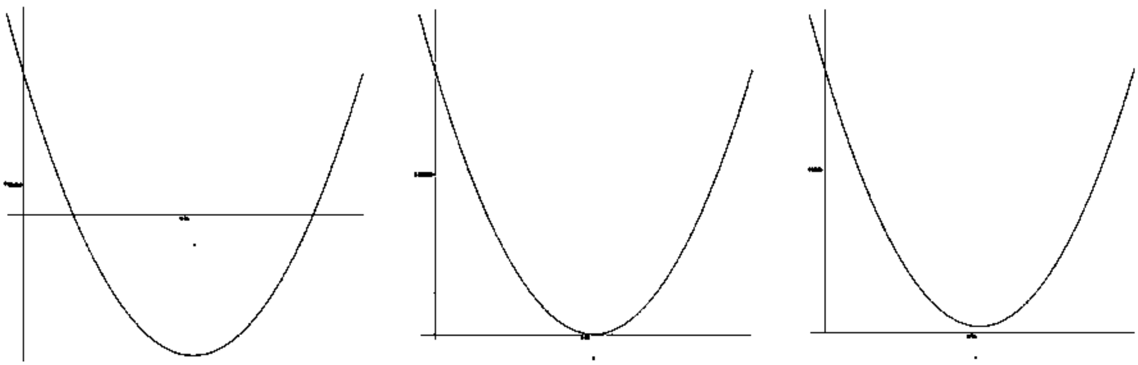
$c = 1/2$ $c = 1$ $c = 3/2$

(2) $f(x) = x - cx(1-x) = x - cx + cx^2 = x(1 - c + cx) = 0 \Leftrightarrow x = 0$ or $x = \frac{c-1}{c}$. Note that when $c = 1$, both equilibrium solutions blend into one. We use the Derivative Test to determine the nature of each equilibrium solution: $f'(x) = 1 - c - 2cx$, so $f'(0) = 1 - c > 0$ if $c < 1$ and $x = 0$ is a source; $f'(0) = 1 - c < 0$ if $c > 1$, so $x = 0$ is a sink. Now $f'\left(\frac{c-1}{c}\right) = 1 - c - 2x\left(\frac{c-1}{c}\right) = 3(1-c) > 0$, so $\frac{c-1}{c}$ is a *source* if $c < 1$ and a *sink* if $c > 1$. Clearly $c = 1$ is the only bifurcation point.

(3)



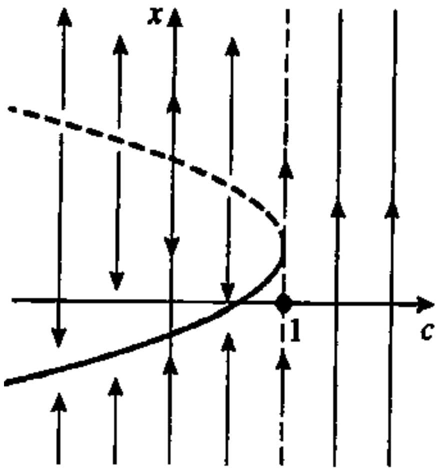
4. (1)



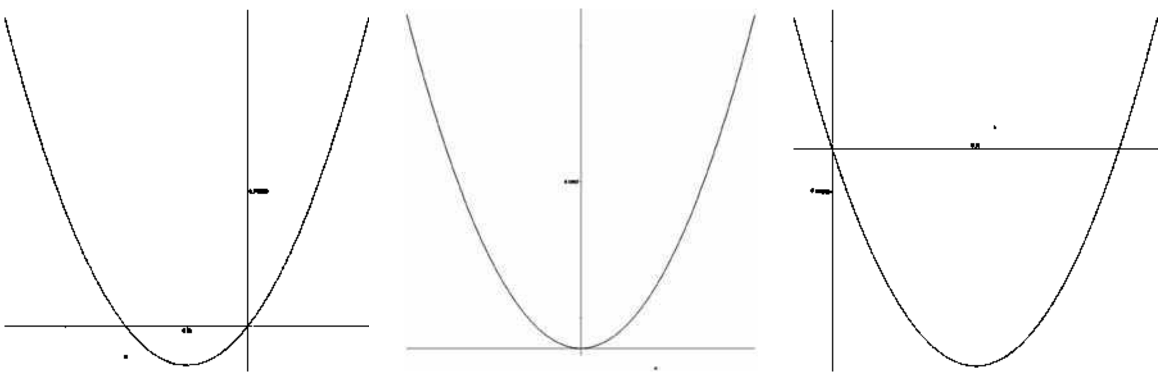
$c = 1/2$ $c = 1$ $c = 3/2$

(2) $f(x) = x^2 - 2x + c = 0 \Leftrightarrow x = \frac{2 \pm \sqrt{4-4c}}{2} = 1 \pm \sqrt{1-c}$. Therefore, we have two equilibrium points if $c < 1$, one equilibrium point if $c = 1$, and no equilibrium point if $c > 1$. Furthermore, $f'(x) = 2x - 2$, so that $f'\left(1 \pm \sqrt{1-c}\right) = 2\left(1 \pm \sqrt{1-c}\right) - 2 = \pm 2\sqrt{1-c}$. Thus, if $c < 1$, the equilibrium point with the positive square root is a source, while the equilibrium solution with the negative square root is a sink. If $c = 1$, then the equilibrium solution is $x = 1$, a node. Since the qualitative nature of the solutions changes when $c = 1$, we see that $c = 1$ is the only bifurcation point.

(3)



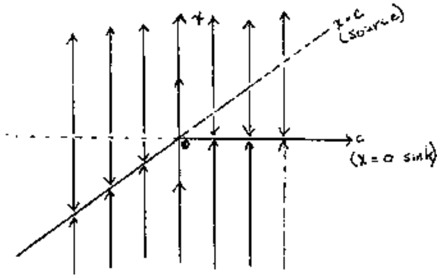
5. (1)



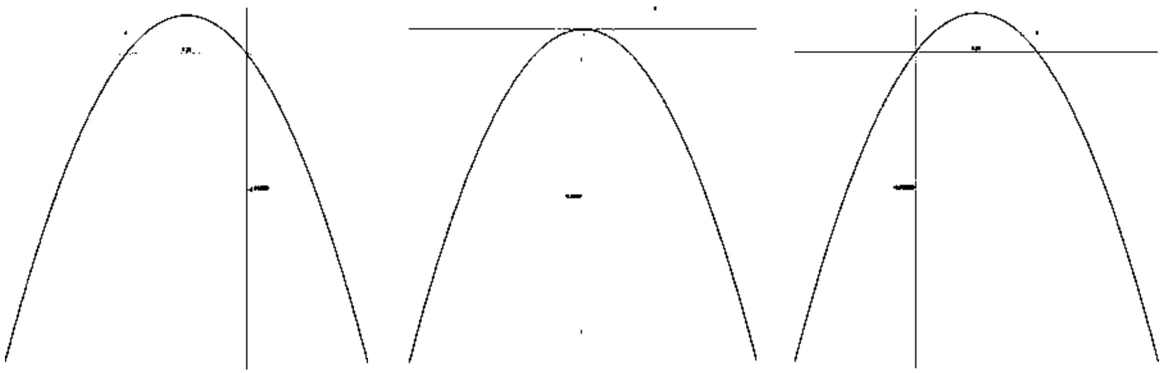
$c = -1/2$ $c = 0$ $c = 3/2$

(2) The bifurcation point is $c = 0$: If $c = 0$, there is only one equilibrium solution, $x = 0$, which is a *node*. If $c > 0$, then $x = 0$ and $x = c$ are equilibrium solutions, with 0 a *sink* and c a *source*; and if $c < 0$, $x = 0$ and $x = c$ are equilibrium solutions, with 0 a *source* and c a *sink*.

(3)



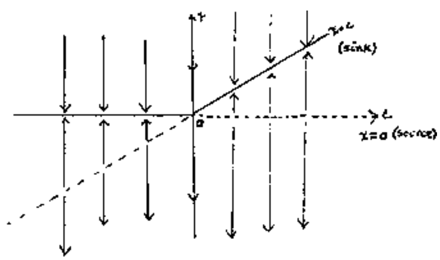
6. (1)



$c = -1/2$ $c = 0$ $c = 3/2$

(2) The bifurcation point is $c = 0$: If $c = 0$, there is only one equilibrium solution, $x = 0$, which is a *node*. If $c > 0$, then $x = 0$ and $x = c$ are equilibrium solutions, with 0 a *node* and c a *sink*; and if $c < 0, x = 0$ and $x = c$ are equilibrium solutions, with 0 a *sink* and c a *source*.

(3)

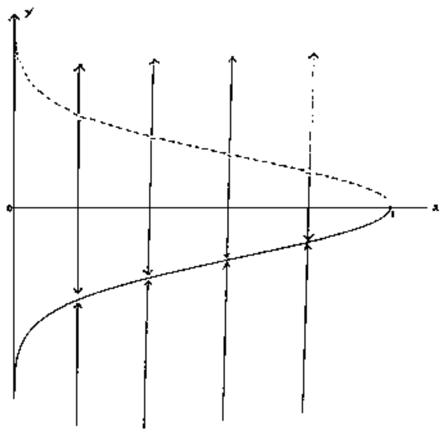


B

1. $f(P) = P(5 - P) - h = 5P - P^2 - h = 0 \Leftrightarrow P = \frac{-5 \pm \sqrt{25-4h}}{-2} = \frac{5 \mp \sqrt{25-4h}}{2}$. The number of equilibrium solutions depends on the discriminant, $25 - 4h$. If $4h < 25$—that is, if $h < 4/25$—then there are *two* nonzero equilibrium solutions. If $4h = 25$, then there is only *one* equilibrium solution, $P = 5/2$. If $4h > 25$, there is *no* equilibrium solution. Clearly, $h = 4/25$ is a bifurcation point. If $h \le 4/25$, we can't have extinction for every initial population because of the nonzero equilibrium solutions, and so we investigate the case $h > 4/25$ more carefully. If $h > 4/25$, the graph of dP/dt is an upside down parabola which doesn't intersect the P-axis. Therefore $dP/dt < 0$ for every solution P, so that every solution—no matter what the initial point $P(0)$—must be decreasing as t increases, eventually reaching zero. This says that $h^* = 25/4$ is the maximum harvest rate beyond which any population will become extinct.

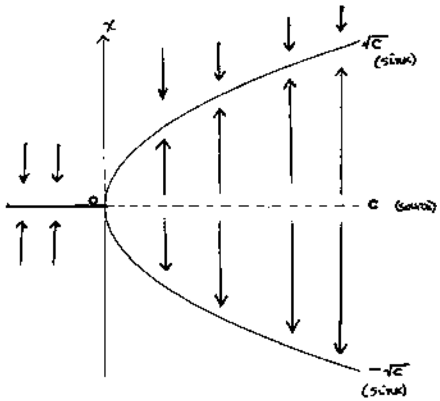
2. We have $x' = \alpha - e^{-x^2}, \alpha > 0$, and $f'(x) = 2xe^{-x^2}$. Now $x' = \alpha - e^{-x^2} = 0 \Leftrightarrow \alpha = e^{-x^2} \Leftrightarrow x = \pm\sqrt{-\ln \alpha}$, where $\ln \alpha \le 0$—that is, where $0 < \alpha \le 1$. If $\boxed{0 < \alpha < 1}$, then $f'\left(-\sqrt{-\ln \alpha}\right) = -2\sqrt{-\ln \alpha}\, e^{-\left(-\sqrt{-\ln \alpha}\right)^2} = -2\alpha\sqrt{-\ln \alpha} < 0$, so $x = -\sqrt{-\ln \alpha}$ is a sink. On the other hand, $f'\left(\sqrt{-\ln \alpha}\right) = 2\sqrt{-\ln \alpha}\, e^{-(-\sqrt{-\ln \alpha})^2} = 2\alpha\sqrt{-\ln \alpha} > 0$, so $x = \sqrt{-\ln \alpha}$ is a source.

   If $\boxed{\alpha = 1}$, then $x = 0$ is the only equilibrium solution. Finally, if $\boxed{\alpha > 1}$, there is no real equilibrium solution. This analysis shows that **$\alpha = 1$ is the only bifurcation point.** Based upon this analysis, we can construct the bifurcation diagram.

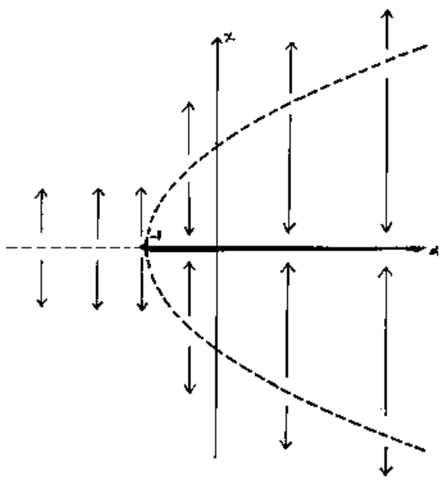


3. We have $f(x) = x(c - x^2)$ and $f'(x) = c - 3x^2$. The equilibrium solutions are $x = 0$ and $x = \pm\sqrt{c}$ if $c > 0$. Then $f'(0) = c$, so $x = 0$ is a sink if $c > 0$ and a source if $c < 0$. Also, $f'\left(-\sqrt{c}\right) = c - 3\left(-\sqrt{c}\right)^2 = -2c < 0$ (since here we must assume $c > 0$), so $x = -\sqrt{c}$ is a sink. Furthermore, $f'\left(\sqrt{c}\right) = c - 3\left(\sqrt{c}\right)^2 = -2c < 0$, so $\sqrt{c}$ is also a sink. If $c = 0$, then $dx/dt = -x^3$ and the only equilibrium solution is $x = 0$, which is a sink: $dx/dt > 0$

if $x < 0$ and $dx/dt < 0$ if $x > 0$. Thus the bifurcation occurs at $c = 0$. The bifurcation diagram follows.

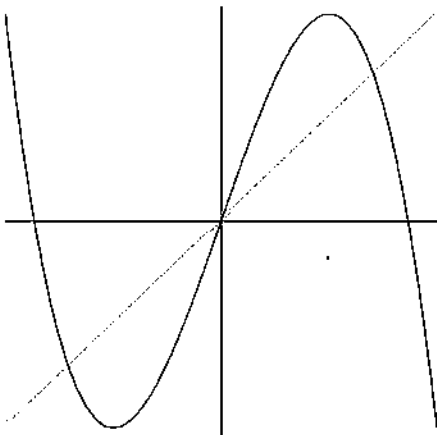


4. We have $f(x) = x(x^2 - 1 - \alpha), \alpha \in \mathbb{R}$, and $f'(x) = 3x^2 - 1 - \alpha$. Thus the equilibrium solutions are $x = 0$ and $x = \pm\sqrt{\alpha + 1}$ for $\alpha > -1$. We see that $f'(0) = -(\alpha + 1)$, so $x = 0$ is a source if $\alpha < -1$ and a sink if $\alpha > -1$. If $\alpha = -1$, then $dx/dt = f(x) = x^3$, so $x = 0$ is a source: $dx/dt = x^3 < 0$ for $x < 0$ and $dx/dt = x^3 > 0$ for $x > 0$. Furthermore, $f'\left(\pm\sqrt{\alpha + 1}\right) = 3\left(\pm\sqrt{\alpha + 1}\right)^2 - 1 - \alpha = 2(\alpha + 1) > 0$ for $\alpha > -1$, so $x = \pm\sqrt{\alpha + 1}$ is a source. Thus the bifurcation point (a *pitchfork bifurcation*) is $\alpha = -1$. The diagram follows.

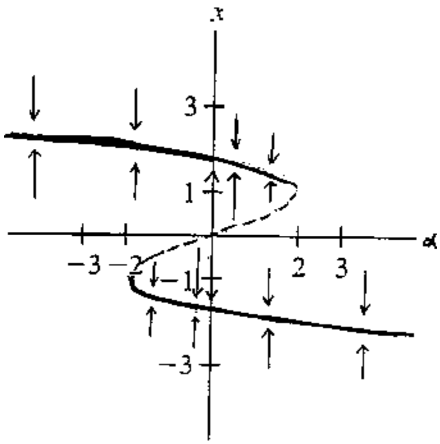


5. This exercise is tricky because $f(x) = 3x - x^3 - \alpha$ and it's difficult to determine the equilibrium solutions algebraically. However, because we are interested in the bifurcation diagram, we can bypass this part of the problem. Instead of solving the equation $3x - x^3 - \alpha = 0$ for x as a function of α, we can graph $\alpha = 3x - x^3$ (showing α as a function of x) and then find the *inverse* of this graph, which represents x as a function of α. This

inverse graph is just the graph of $\alpha = 3x - x^3$ reflected across the line $\alpha = x$. Thus, given the graphs of the cubic polynomial $\alpha = 3x - x^3$ and $\alpha = x$



we get the bifurcation diagram by reflection:



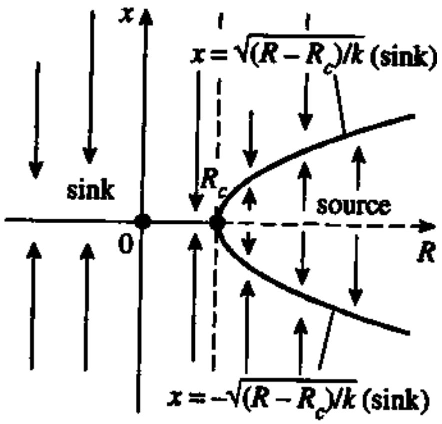
There are two "saddle-node bifurcations, one at $\alpha = -2$ and the other at $\alpha = 2$. We can also see that there is one equilibrium point for $\alpha < -2$ and one for $\alpha > 2$. For α between -2 and 2, there are three equilibrium points.

C

1. a. $\dot{x} = f(x) = (R - R_c)x - kx^3 = 0 \Leftrightarrow x\left[(R - R_c) - kx^2\right] = 0 \Leftrightarrow x = 0$ or $x = \pm\sqrt{(R - R_c)/k}$. If $R < R_c$, then $R - R_c < 0$, so that there is only one equilibrium solution, $x = 0$. Since $f'(0) = (R - R_c) - 3k(0)^2 = R - R_c < 0$, we see that $x = 0$ is a *sink*.

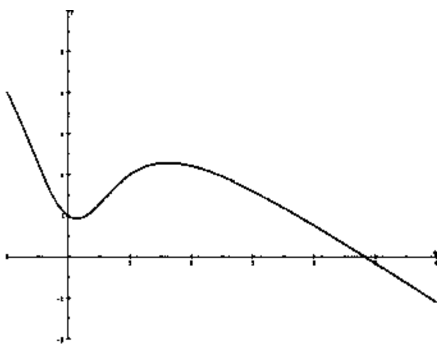
b. If $R > R_c$, then $R - R_c > 0$ and we have the three equilibrium solutions $x = 0$, $x = \sqrt{(R-R_c)/k}$, and $x = -\sqrt{(R-R_c)/k}$. Then $f'(0) = (R-R_c) - 3k(0)^2 = R - R_c > 0$, showing that $x = 0$ is a source. Also, $f'\left(\pm\sqrt{(R-R_c)/k}\right) = (R-R_c) - 3k\left(\sqrt{(R-R_c)/k}\right)^2 = -2(R-R_c) < 0$, so that $\sqrt{(R-R_c)/k}$ and $-\sqrt{(R-R_c)/k}$ are sinks.

c.

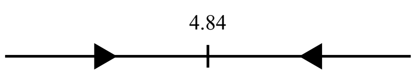


The bifurcation point $R = R_c$ is a *sink*.

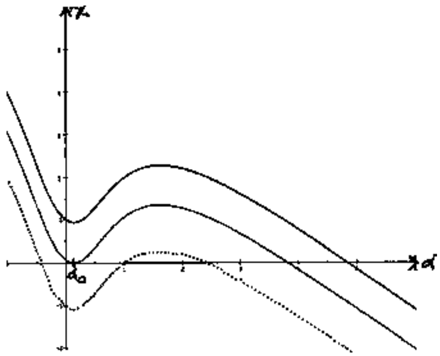
2. a. If $\alpha = 1$, we have $dx/dt = f(x) = 1 - x + 4x^2/(1+x^2)$. Now the graph of $f(x)$ shows that there is only one equilibrium solution, $x \approx 4.84$.



Furthermore, the graph shows that $f(x) > 0$ for $x < 4.84$ and $f(x) < 0$ for $x > 4.84$. This information gives us the phase portrait



b. Looking at graphs of $f(x)$ with various values of α (and playing with the **solve** command on a graphing calculator or CAS), we can estimate that a bifurcation occurs when $\alpha = \alpha_0 = 0.063525\ldots \approx 0.064$.



These graphs suggest that when $\alpha > \alpha_0$ there is only one equilibrium solution, a sink. When $\alpha = \alpha_0$, there are two distinct equilibrium solutions, one a node and one a sink. Finally, when $\alpha < \alpha_0$, there are three equilibrium solutions—a sink, a source, and a sink, reading from left to right.

2.8 EXISTENCE AND UNIQUENESS OF SOLUTIONS

Usually I don't cover the material on existence and uniqueness as extensively as indicated in this section. The ideas are important, but fortunately (as indicated in Example 2.8.3) most of the equations seen in basic science and engineering courses have unique solutions, given appropriate initial conditions.

I believe that the particular existence and uniqueness theorem I give is the easiest to understand and to use. Appendix A has a brief explanation of partial derivatives if necessary.

Exercise C6, although found in an old ODE text by Buck, is the kind of problem that a modern differential equations course should pose. The student should be encouraged to *think*, rather than just solve equations.

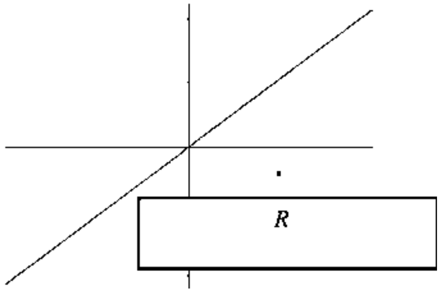
An interesting problem that I sometimes include in my class discussion is to show that the IVP $x' = x^2 + t, x(0) = 0$, has no solution defined on the whole interval $(0, 3)$. (Note that the equation is neither separable nor linear. In fact, it has no solution in terms of elementary functions.) The slope field for this equation seems to indicate that any solution curve defined on the interval $(0, 3)$ increases without bound. (This is the phenomenon that can be described as "blowing up in finite time.") The solution has the bonus of acquainting students with *proof by contradiction*: Suppose that there *is* a solution $x(t)$ of the IVP that is defined on the entire interval $(0, 3)$. Form the function $z(t) = \arctan x(t), 0 \leq z(t) < \pi/2$. Note that $z(0) =$

$\arctan x(0) = \arctan 0 = 0$. Then the Chain Rule gives $z'(t) = \frac{1}{1+[x(t)]^2} \cdot x'(t)$. In a way that may be familiar to students from previous courses, construct a right triangle with one acute angle equal to $\arctan x(t)$. Then $\tan z = \tan(\arctan x(t)) = x(t)$, $\cos z = \cos(\arctan x(t)) = \frac{1}{\sqrt{x^2+1}}$, $\cos^2 z = \frac{1}{x^2+1}$, and $\cos^{-2} z = x^2 + 1$. Thus $(\cos^{-2} z)z' = x'(t) = x^2 + t = \tan^2 z + t$, with $0 \le z(t) < \pi/2$. Multiplying the last equation by $\cos^2 z$, we get $z' = \cos^2 z \cdot \frac{\sin^2 z}{\cos^2 z} + t\cos^2 z = \sin^2 z + t\cos^2 z = \sin^2 z + \cos^2 z + (t-1)\cos^2 z = 1 + (t-1)\cos^2 z \ge 1$ for $t \ge 1$. Because we have $z(0) = 0$ and $z' \ge 1$ for $t \ge 1$, we see that $z(t) \ge t - 1$. By the FTC and some elementary integral comparisons, $z(t) = \int_0^t z'(t)dt = \int_0^1 z'(t)dt + \int_1^t z'(t)dt = z(1) + \int_1^t z'(t)dt \ge \int_1^t z'(t)dt \ge \int_1^t 1dt = t - 1$.

Letting $t = 2.9$ in this last inequality, we conclude that $z(2.9) \ge 2.9 - 1 = 1.9$, which is greater than $\pi/2 \approx 1.5708$—a contradiction of the fact that we assumed $0 \le z(t) < \pi/2$. *Therefore, a solution $x(t)$ can't exist.*

A

1. $f(t, x) = \frac{1}{x}$ and $\frac{\partial f}{\partial x} = -\frac{1}{x^2}$ are not continuous where $x = 0$. So, for example, take any rectangle centered at $(0, 3)$ that avoids the t-axis ($x = 0$).
2. $f(t, y) = \frac{5}{4}y^{1/5}$ and $\frac{\partial f}{\partial y} = \frac{1}{4}y^{-4/5}$, so that the partial derivative is not continuous when $y = 0$. Therefore, the IVP has no unique solution through $(t, y) = (0, 0)$. For example, both $y = t^{\frac{5}{4}}$ and $y = -t^{\frac{5}{4}}$ are solutions passing through the origin.
3. $f(t, x) = \frac{x}{t}$ and $\frac{\partial f}{\partial x} = \frac{1}{t}$ are not continuous when $t = 0$—that is, at any point of the x-axis. There is no rectangle R containing the origin that does not also include points of the x-axis, where $t = 0$.
4. $f(t, y) = -\frac{t}{y}$ and $\frac{\partial f}{\partial y} = \frac{t}{y^2}$ are continuous in any rectangle that does not include $y = 0$. Because the initial point we are given is $(0, 0.2)$, we can construct a rectangle centered at $(0, 0.2)$ such that its lower side intersects the y-axis at some value $d > 0$. For example, consider the square with vertices $(0.1, 0.1)$, $(0.1, 0.3)$, $(-0.1, 0.3)$, and $(-0.1, 0.1)$.
5. $f(t, y) = \frac{t}{1+t+y}$ and $\frac{\partial f}{\partial y} = -\frac{t}{(1+t+y)^2}$ fail to be continuous at those points (t, y) for which $1 + t + y = 0$—that is, at points on the straight line $y = -t - 1$. But the initial condition specifies the point $(-2, 1)$, which lies on this line; and any rectangle that includes $(-2, 1)$ also includes infinitely many points on the line $y = -t - 1$. Clearly, there is no rectangle R satisfying the requirements of the Existence and Uniqueness Theorem.
6. $f(x) = \tan x = \frac{\sin x}{\cos x}$ and $\frac{\partial f}{\partial x} = f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$ are not continuous where $\cos x = 0$. Unfortunately, $x = \frac{\pi}{2}$ is such a point, so that there is no rectangle guaranteeing us existence and uniqueness.
7. We can write the equation in the form $\frac{dy}{dt} = \frac{1-y}{1+t} = f(t, y)$. Then $\frac{\partial f}{\partial y} = -\frac{1}{1+t}$. The rectangle R can be any rectangle in the t-y plane that does not contain $t = -1$. For example, consider the square with vertices $(1.5, 1)$, $(-0.5, 1)$, $(-0.5, -1)$, and $(1.5, -1)$.
8. $f(x, y) = \frac{x+y}{x-y}$ and $\frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}$ fail to be continuous at those points (t, y) for which $y = x$. Thus R can be any rectangle that does not include points on the line $y = x$.



9. This is a separable equation whose unique solution (using the initial condition) is $y = \tan x$. A look at the graph of this tangent function reveals that the function's domain is $I = (-\pi/2, \pi/2)$, an interval of length π.
10. $f(t, y) = t(1 + y)$ and $\frac{\partial f}{\partial y} = t$ are both continuous everywhere in the t-y plane. Therefore, there is a rectangle R centered at $(0, -1)$ such that the given IVP has a unique solution in R. But clearly $y(t) \equiv -1$ is a solution in the rectangle, and so must be the *only* solution.
11. Separating the variables, we get $x^{-2/3}dx = dt$, so that $3x^{1/3} = t + C, x^{1/3} = t/3 + K$, and $x(t) = (t/3 + K)^3$. The initial condition implies that $K = x_0$, or $K = \sqrt[3]{x_0}$. Therefore, the solution of the IVP is $x(t) = \left(\frac{t}{3} + \sqrt[3]{x_0}\right)^3$. Note that $f(t, x) = x^{2/3}$ and $\frac{\partial f}{\partial x} = \frac{2}{3\sqrt[3]{x}}$, so that the partial derivative is not continuous at $x = 0$. The initial condition of Example 2.8.2 is $x(0) = 0$, so that we don't expect uniqueness in this case. In the current exercise, both f and $\frac{\partial f}{\partial x}$ are continuous at $(0, x_0)$ if $x_0 < 0$, so that we are guaranteed existence and uniqueness on some t-interval I.

B

1. a. $f(Q) = |Q - 1|$ is continuous everywhere, but $\frac{\partial f}{\partial Q}$ is not defined at $Q = 1$ (loosely, because of the sharp point in the graph of $f(Q)$ at 1). Therefore, the conditions of the Existence and Uniqueness Theorem do not hold.
   b. The constant function $Q \equiv 1$ is a solution because $Q' = 0 = |Q - 1|$ and $Q(0) = 1$. This solution is in fact unique, showing that the Existence and Uniqueness Theorem provides *sufficient* conditions that are not *necessary*.
   c. Consider the two possibilities separately:
   If $Q(x) > 1$, then $Q' = |Q - 1|$ becomes $Q' = Q - 1$, with $Q \equiv 1$ if $Q(0) = 1$. If $Q(x) < 1$, then $Q' = |Q - 1|$ becomes $Q' = 1 - Q$, with $Q \equiv 1$ if $Q(0) = 1$.
2. a. We can write the equation as follows:

$$\dot{y} = \begin{cases} \sqrt{y} + k & \text{if} \quad y > 0 \\ \sqrt{-y} + k & \text{if} \quad y < 0 \end{cases}$$

   Separating variables in each equation and making the appropriate substitution $u = \sqrt{y} - k$ or $u = \sqrt{-y} - k$, we can integrate to find the implicit solution

$2\sqrt{y} - 2k\ln\left(\sqrt{y}+k\right) = t + C$ if $y > 0$ and $2\sqrt{|y|} - 2k\ln\left(\sqrt{|y|}+k\right) = t + C$ if $y < 0$.

b. Because $k > 0$, there is a unique solution for *any* initial condition even though $\frac{\partial f}{\partial y}$ does not exist at $y = 0$. As in Exercise B1, the point is that the Existence and Uniqueness Theorem provides *sufficient* conditions that are not *necessary*. Note that $y \equiv 0$ is not a solution of the equation.

c. If $k < 0$, the equation has a unique solution for any initial condition. When $k = 0$, we have the solution given in part (a) as well as $y \equiv 0$, so that the IVP with initial condition $y(0) = 0$ has *no* unique solution.

3. $f(x,y) = 2x - 2\sqrt{x^2 - y}$ and $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2-y}}$. The partial derivative is not continuous at any point (x, y) on the parabola $y = x^2$—in particular at the point $(1, 1)$—so that the conditions of the Existence and Uniqueness Theorem are not satisfied and uniqueness is not guaranteed.

4. a. $f(x,y) = \cos x - x^2y^3$ and $\frac{\partial f}{\partial y} = -3x^2y^2$ are continuous at every point of the x-y plane. Therefore, given a point (x_0, y_0), the equation does have a unique solution passing through (x_0, y_0).

b. Here's how *Maple* handled the equation:

```
> with(DEtools):

> Eq:=diff(y(x),x)+x^2*(y(x))^2=cos(x);
```

$$Eq := \left(\frac{\partial}{\partial x}y(x)\right) + x^2y(x)^2 = \cos(x)$$

```
> dsolve(Eq,y(x));

>
```

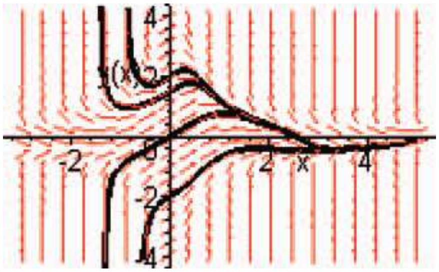
The last line indicates *Maple*'s response—a blank. This says that *Maple* could not come up with a solution. The moral is that technology doesn't have all the answers. We know from part (a) that every IVP that involves this equation has a unique solution, but we can't seem to find it! However, the following commands (using a numerical method—see Chapter 3—to calculate points on a solution curve) enable us to see some solution curves corresponding to four different initial conditions:

```
> with(DEtools):

> DEplot(diff(y(x),x)+x^2*(y(x))^3=cos(x),y(x),

x=-3..5,{[0,2],[0,-2],[-1,1],[0,0]},

  y=-4..4,stepsize=.01,linecolor=black);
```



C

1. Here $f(x, y) = P(x)y^2 + Q(x)y$ and $\frac{\partial f}{\partial y} = 2P(x)y + Q(x)$. Because P and Q are polynomials (continuous everywhere), f and $\frac{\partial f}{\partial y}$ are also continuous everywhere in the x-y plane. The conditions of the Existence and Uniqueness Theorem are satisfied, and so we expect to find an interval $I = (2 - h, 2 + h)$ centered at $x = 2$ such that the IVP has a unique solution on I.
2. $f(x) = (\alpha - x)(\beta - x)$ and $\frac{\partial f}{\partial x} = f'(x) = 2x - (\alpha + \beta)$ are continuous at every point (t, x), so that if we are given any initial point (t_0, x_0), we can find a unique solution passing through (t_0, x_0).
3. $f(P) = kP(b - P)$ and $\frac{\partial f}{\partial P} = kb - 2kP$ are continuous at every point (t, P), so that any IVP involving the logistic equation has a unique solution. If a solution near $P \equiv b$ were to equal the equilibrium solution—that is, if another solution curve intersects the horizontal line $P \equiv b$ at the point (t^*, b)—then we would have *two* solutions of the IVP $\frac{dP}{dt} = kP(b-P)$, $P(t^*) = b$.
4. Every point at which two curves intersect is a point of nonuniqueness, contradicting the Existence and Uniqueness Theorem which should apply when f is a polynomial function. (See the solution to Exercise C3 for a similar argument.)
5. We are given the IVP $y' = y$, $y(0) = 1$, which has a unique solution.

   a. Consider $Y(t) = y(t)y(-t)$. Then $Y'(t) = y(t)[-y'(-t)] + y'(t)y(-t) = -y(t)y'(-t) + y'(t)y(-t) = -y(t)y(-t) + y(t)y(-t) = 0$, which implies that $Y(t)$ is a constant function. Since $Y(0) = y(0)y(0) = 1$, we see that the constant must be 1—that is, $Y(t) = y(t)y(-t) = 1$ for all values of t.
   b. By part (a), $y(t)$ can never be zero. We are given that $y(0)$ is positive. If $y(t^*)$ were negative for some t^*, by the Intermediate Value Theorem $y(\hat{t})$ would be zero for some $\hat{t}$ between 0 and t^*, which cannot happen. Thus $y(t) > 0$ for all values of t.
   c. Let t_2 be an arbitrary fixed real number and consider the function $Q(t) = \frac{y(t+t_2)}{y(t)}$. Notice that $Q(0) = y(t_2)$. Then $Q'(T) = \frac{y(t)y'(t+t_2) - y(t+t_2)y'(t)}{[y(t)]^2}$, which equals zero because $y' = y$. Therefore Q is constant, so it is equal to $y(t_2)$ for all values of t.

6. If the equation does not have a solution, this may indicate that the reaction cannot take place. However, our confidence in this conclusion must be proportional to our belief that the differential equation provides an accurate description of our experiment. If the

reaction *does* take place, we should conclude that the differential equation model is not accurate. On the other hand, if the equation has a solution, this does not guarantee that the reaction does take place, because the model may not be an accurate description of physical reality. For example, in Chapter 4 we'll see a model of a spring-mass system that predicts the mass will never stop bobbing up and down, which certainly is not an accurate description of reality.