CHAPTER 2

2.1 Show that for an electromagnetic wave traveling through a dielectric ($m_1 = n_1$), impinging on the interface with another, optically less dense dielectric ($n_2 < n_1$), light of any polarization is totally reflected for incidence angles larger than $\theta_c = \sin^{-1}(n_2/n_1)$.
Hint: Use equations (2.105) with $k_2 = 0$.

Solution
Equations (2.105) become for $k_2 = 0$,

$$\eta_0 \, n_1 \, \sin\theta_1 = w_t' \, \sin\theta_2,$$
$$w_t'^2 - w_t''^2 = \eta_0^2 \, n_2^2,$$
$$w_t' \, w_t'' \, \cos\theta_2 = 0.$$

The last of these relations dictates that either $w_t'' = 0$ or $\cos\theta_2 = 0$ ($w_t' = 0$ is not possible since—from the first relation—this would imply $\eta_0 \, n_1 \, \sin\theta_1 = 0$ which is known not to be true).

$w_t'' = 0$: Substituting this into the second relation leads to $w_t' = \eta_0 \, n_2$, and the first leads to $n_1 \, \sin\theta_1 = n_2 \, \sin\theta_2$.
Since $n_1 > n_2$ this is a legitimate solution only for $\sin\theta_1 \le (n_2/n_1)$, or $\theta_1 \le \theta_c = \sin^{-1}(n_2/n_1)$.

$\cos\theta_2 = 0$ ($\theta_2 = \pi/2$): Substituting the first relation into the second gives

$$w_t'' = \eta_0 \, \sqrt{n_1^2 \, \sin^2\theta_1 - n_2^2},$$

i.e., a legitimate nonzero solution for $n_1^2 \, \sin^2\theta_1 - n_2^2 \ge 0$ or $\theta_1 \ge \theta_c$. Inspection of the reflection coefficients, equations (2.109), shows that

$$\widetilde{r}_\parallel = \frac{i n_1^2 \, w_t'' + n_2^2 \, w_i' \, \cos\theta_1}{i n_1^2 \, w_t'' - n_2^2 \, w_i' \, \cos\theta_1}, \quad \widetilde{r}_\perp = \frac{w_i' \, \cos\theta_1 + i w_t''}{w_i' \, \cos\theta_1 - i w_t''}$$

Since, in both reflection coefficients, there are no sign changes within the real and imaginary parts, it follows readily that

$$\rho_\parallel = \widetilde{r}_\parallel \widetilde{r}_\parallel^* = \rho_\perp = \widetilde{r}_\perp \widetilde{r}_\perp^* = 1.$$

2.2 Derive equations (2.109) using the same approach as in the development of equations (2.89) through (2.92). Hint: Remember that within the absorbing medium, $\mathbf{w} = \mathbf{w}' - i\mathbf{w}'' = w'\hat{\mathbf{s}} - iw''\hat{\mathbf{n}}$; this implies that $\mathbf{E}_0$ is *not* a vector normal to $\hat{\mathbf{s}}$. It is best to assume $\mathbf{E}_0 = E_\parallel\hat{\mathbf{e}}_\parallel + E_\perp\hat{\mathbf{e}}_\perp + E_s\hat{\mathbf{s}}$.

Solution
Inside the absorbing medium $\mathbf{w}_t = \mathbf{w}_t' - i\mathbf{w}_t'' = w_t'\hat{\mathbf{s}}_t - iw_t''\hat{\mathbf{n}}$, and the electric field vector does not lie in a plane normal to $\hat{\mathbf{s}}$. Thus, we assume a general three-dimensional representation, or

$$\mathbf{E}_0 = E_\parallel\hat{\mathbf{e}}_\parallel + E_\perp\hat{\mathbf{e}}_\perp + E_s\hat{\mathbf{s}}.$$

Following the development for nonabsorbing media, equations (2.77) through (2.88), then leads to

$$\nu\mu\mathbf{H}_0 = \mathbf{w}\times\mathbf{E}_0 = (w'\hat{\mathbf{s}} - iw''\hat{\mathbf{n}})\times(E_\parallel\hat{\mathbf{e}}_\parallel + E_\perp\hat{\mathbf{e}}_\perp + E_s\hat{\mathbf{s}}).$$

This formulation is valid for the transmitted wave, but also for the incident wave ($w_i'' = 0$) and reflected wave ($w_r'' = 0$, $w_r' = -w_i'$). The contribution from E_s vanishes for incident and reflected wave. Using the same vector relations as given for the nonabsorbing media interface, one obtains

$$\nu\mu\mathbf{H}_0 = w'(E_\parallel\hat{\mathbf{e}}_\perp - E_\perp\hat{\mathbf{e}}_\parallel) - iw''\,(E_\parallel\hat{\mathbf{e}}_\perp\cos\theta - E_\perp\hat{\mathbf{t}} + E_s\hat{\mathbf{e}}_\perp\sin\theta).$$

For the interface condition (with $\nu\mu$ the same everywhere)

$$\nu\mu H_0\times\hat{\mathbf{n}} = w'(E_\parallel\hat{\mathbf{t}} + E_\perp\hat{\mathbf{e}}_\perp\cos\theta) - iw''\,(E_\parallel\cos\theta\hat{\mathbf{t}} + E_\perp\hat{\mathbf{e}}_\perp + E_s\sin\theta\hat{\mathbf{t}}).$$

Thus, from equation (2.78)

$$\begin{aligned} &w_i'(E_{i\parallel}\hat{\mathbf{t}} + E_{i\perp}\hat{\mathbf{e}}_\perp\cos\theta_1) - w_i'(E_{r\parallel}\hat{\mathbf{t}} + E_{r\perp}\hat{\mathbf{e}}_\perp\cos\theta_1)\\ &\quad = w_t'(E_{t\parallel}\hat{\mathbf{t}} + E_{t\perp}\hat{\mathbf{e}}_\perp\cos\theta_2) - iw_t''(E_{t\parallel}\cos\theta_2\hat{\mathbf{t}} + E_{t\perp}\hat{\mathbf{e}}_\perp + E_{ts}\sin\theta_2\hat{\mathbf{t}})\end{aligned}$$

or

$$w_i'(E_{i\parallel} - E_{r\parallel}) = (w_t' - iw_t''\cos\theta_2)\,E_{t\parallel} - iw_t''\sin\theta_2 E_{ts} \tag{2.2-A}$$
$$w_i'(E_{i\perp} - E_{r\perp})\cos\theta_1 = (w_t'\cos\theta_2 - iw_t'')\,E_{t\perp} \tag{2.2-B}$$

Similarly, from equation (2.77),

$$\mathbf{E}_0\times\hat{\mathbf{n}} = (E_\parallel\hat{\mathbf{e}}_\parallel + E_\perp\hat{\mathbf{e}}_\perp + E_s\hat{\mathbf{s}})\times\hat{\mathbf{n}} = -E_\parallel\hat{\mathbf{e}}_\perp\cos\theta + E_\perp\hat{\mathbf{t}} - E_s\hat{\mathbf{e}}_\perp\sin\theta$$

and

$$(E_{i\parallel} + E_{r\parallel})\cos\theta_1 = E_{t\parallel}\cos\theta_2 + E_{ts}\sin\theta_2 \tag{2.2-C}$$
$$E_{i\perp} + E_{r\perp} = E_{t\perp} \tag{2.2-D}$$

These four equations have 5 unknowns ($E_{r\parallel}$, E_{rs}, $E_{t\parallel}$, $E_{t\perp}$, and E_{ts}), and an additional condition is needed, e.g., equation (2.23) or equation (2.64). Choosing equation (2.23) we obtain, inside the absorbing medium,

$$\begin{aligned}\mathbf{w}\cdot\mathbf{E}_0 = 0 &= (w_t'\hat{\mathbf{s}} - iw_t''\hat{\mathbf{n}})\cdot(E_{t\parallel}\hat{\mathbf{e}}_{t\parallel} + E_{t\perp}\hat{\mathbf{e}}_\perp + E_{ts}\hat{\mathbf{s}})\\ &= w_t'E_{ts} + iw_t''(E_{t\parallel}\sin\theta_2 - E_{ts}\cos\theta_2).\end{aligned} \tag{2.2-E}$$

Eliminating $E_{t\perp}$ from equations (2.2-B) and (2.2-D), with $\widetilde{r}_\perp = E_{r\perp}/E_{i\perp}$, gives

$$w_i'(1-\widetilde{r}_\perp)\cos\theta_1 = (w_t'\cos\theta_2 - iw_t'')(1+\widetilde{r}_\perp),$$

or

$$\boxed{\widetilde{r}_\perp = \frac{w_i'\cos\theta_1 - (w_t'\cos\theta_2 - iw_t'')}{w_i'\cos\theta_1 + (w_t'\cos\theta_2 - iw_t'')},}$$

which is identical to equation (2.109).

Now, eliminating E_{ts} from equations (2.2-A) and (2.2-C) [multiplying equation (2.2-C) by iw_t'' and adding]:

$$w_i'(E_{i\parallel} - E_{r\parallel}) + iw_t'' \cos\theta_1(E_{i\parallel} + E_{r\parallel}) = w_t' E_{t\parallel}. \tag{2.2-F}$$

Eliminating E_{ts} from equations (2.2-C) and (2.2-E) leads to

$$\begin{aligned} E_{ts} &= \frac{-iw_t'' E_{t\parallel} \sin\theta_2}{w_t' - iw_t'' \cos\theta_2} \\ (E_{i\parallel} + E_{r\parallel}) \cos\theta_1 &= E_{t\parallel}\left[\cos\theta_2 - \frac{iw_t'' \sin^2\theta_2}{w_t' - iw_t'' \cos\theta_2}\right] = E_{t\parallel}\frac{w_t' \cos\theta_2 - iw_t''}{w_t' - iw_t'' \cos\theta_2}. \end{aligned}$$

Using this to eliminate $E_{t\parallel}$ from equation (2.2-F), with $\widetilde{r}_\parallel = E_{r\parallel}/E_{i\parallel}$, gives

$$\begin{aligned} & w_i'(1 - \widetilde{r}_\parallel) + iw_t'' \cos\theta_1(1 + \widetilde{r}_\parallel) = w_t' \cos\theta_1(1 + \widetilde{r}_\parallel)\frac{w_t' - iw_t'' \cos\theta_2}{w_t' \cos\theta_2 - iw_t''} \\ & w_i'(w_t' \cos\theta_2 - iw_t'')(1 - \widetilde{r}_\parallel) \\ & \quad = [w_t'(w_t' - iw_t'' \cos\theta_2) - iw_t''(w_t' \cos\theta_2 - iw_t'')] \cos\theta_1(1 + \widetilde{r}_\parallel) \\ & \quad = \eta_0^2 m_2^2 \cos\theta_1(1 + \widetilde{r}_\parallel), \end{aligned}$$

$$\boxed{\widetilde{r}_\parallel = \frac{w_i'(w_t' \cos\theta_2 - iw_t'') - \eta_0^2 m_2^2 \cos\theta_1}{w_i'(w_t' \cos\theta_2 - iw_t'') + \eta_0^2 m_2^2 \cos\theta_1},}$$

which is the same as equation (2.109).

It is a simple matter to show that other conditions give the same result. For example, from equation (2.64)

$$n_1^2(E_{i\parallel}\hat{\mathbf{e}}_{i\parallel} \cdot \hat{\mathbf{n}} + E_{r\parallel}\hat{\mathbf{e}}_{r\parallel} \cdot \hat{\mathbf{n}}) = m_2^2(E_{t\parallel}\hat{\mathbf{e}}_{t\parallel} \cdot \hat{\mathbf{n}} + E_{ts}\hat{\mathbf{s}} \cdot \hat{\mathbf{n}})$$

or

$$n_1^2(E_{i\parallel} - E_{r\parallel}) \sin\theta_1 = m_2^2(E_{t\parallel} \sin\theta_2 - E_{ts} \cos\theta_2), \text{ etc.}$$

2.3 Find the normal spectral reflectivity at the interface between two absorbing media. [Hint: Use an approach similar to the one that led to equations (2.89) and (2.90), keeping in mind that all wave vectors will be complex, but that the wave will be homogeneous in both media, i.e., all components of the wave vectors are colinear with the surface normal].

Solution

Equations (2.19) and (2.20) remain valid for incident, reflected and transmitted waves, with $\mathbf{w} = \mathbf{w}' - i\mathbf{w}'' = (w' - iw'')\,\hat{\mathbf{n}}$ for all three cases. From equation (2.31) $\mathbf{w}\cdot\mathbf{w} = (w' - iw'')^2\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = \eta_0^2 m^2$ it follows that $w' - iw'' = \pm\eta_0\, m$. Thus

$$\begin{aligned} w_i' - iw_i'' &= \eta_0\, m_1, \\ w_r' - iw_r'' &= -\eta_0\, m_1 \quad \text{(reflected wave is moving in a direction of } -\hat{\mathbf{n}}\text{)}, \\ w_t' - iw_t'' &= \eta_0\, m_2. \end{aligned}$$

From equations (2.23) and (2.24), it follows that the electric and magnetic field vectors are normal to $\hat{\mathbf{n}}$, i.e., tangential to the surface, say $\mathbf{E}_0 = E_0\hat{\mathbf{t}}$. Then, from equation (2.77)

$$(E_i + E_r)\,\hat{\mathbf{t}}\times\hat{\mathbf{n}} = E_t\,\hat{\mathbf{t}}\times\hat{\mathbf{n}},$$

or

$$E_i + E_r = E_t$$

From equation (2.25) $\nu\mu\mathbf{H}_0 = \mathbf{w}\times\mathbf{E}_0 = (w' - iw'')\,E\,\hat{\mathbf{n}}\times\hat{\mathbf{t}}$, and from equation (2.78)

$$n_1(E_i - E_r) = m_2\, E_t.$$

Substituting for E_t and dividing by E_i, with $\widetilde{r} = E_r/E_i$:

$$m_1(1 - \widetilde{r}) = m_2(1 + \widetilde{r})$$

or

$$\widetilde{r} = \frac{m_1 - m_2}{m_1 + m_2}$$

and

$$\rho_n = \widetilde{r}\widetilde{r}^* = \frac{(m_1 - m_2)(m_1 - m_2)^*}{(m_1 + m_2)(m_1 + m_2)^*} = \left|\frac{(n_1 - n_2) + i(k_1 - k_2)}{(n_1 + n_2) + i(k_1 + k_2)}\right|^2$$

$$\boxed{\rho_n = \frac{(n_1 - n_2)^2 + (k_1 - k_2)^2}{(n_1 + n_2)^2 + (k_1 + k_2)^2}}$$

2.4 A circularly polarized wave in air is incident upon a smooth dielectric surface ($n = 1.5$) with a direction of 45° off normal. What are the normalized Stokes' parameters before and after the reflection, and what are the degrees of polarization?

Solution
From the definition of Stokes' parameters the incident wave has degrees of polarization

$$\boxed{\frac{Q_i}{I_i} = \frac{U_i}{I_i} = 0, \qquad \frac{V_i}{I_i} = \pm 1,}$$

the sign giving the handedness of the circular polarization. With $E_{r\|} = E_{i\|} r_\|$ and $E_{r\perp} = E_{i\perp} r_\perp$, from equations (2.50) through (2.53):

$$I_r = E_{i\|} E^*_{i\|} r^2_\| + E_{i\perp} E^*_{i\perp} r^2_\perp = E_{i\|} E^*_{i\|} (\rho_\| + \rho_\perp) = \frac{1}{2}(\rho_\| + \rho_\perp) I_i$$

Since $E_{i\|} E^*_{i\|} = E_{i\perp} E^*_{i\perp}$ [from equation (2.51)] and $\rho = r^2$.
Similarly,

$$\begin{aligned}
Q_r &= E_{i\|} E^*_{i\|} r^2_\| - E_{i\perp} E^*_{i\perp} r^2_\perp = E_{i\|} E^*_{i\|} (\rho_\| - \rho_\perp) \\
U_r &= E_{i\|} E^*_{i\perp} r_\| r_\perp + E_{i\perp} E^*_{i\|} r_\perp r_\| = U_i r_\| r_\perp = 0 \\
V_r &= i(E_{i\|} E^*_{i\perp} - E_{i\perp} E^*_{i\|})\, r_\| r_\perp = V_i r_\| r_\perp \\
\frac{Q_r}{I_r} &= \frac{\rho_\| - \rho_\perp}{\rho_\| + \rho_\perp}, \quad \frac{V_r}{I_r} = \frac{2 r_\| r_\perp}{\rho_\| + \rho_\perp} \frac{V_i}{I_i}.
\end{aligned}$$

From Snell's law

$$\sin\theta_2 = \frac{\sin\theta_1}{n_2}; \ \cos\theta_2 = \sqrt{1 - \frac{\sin^2\theta_1}{n_2^2}} = \sqrt{1 - \frac{0.5}{1.5^2}} = \sqrt{\frac{7}{9}},$$

and from equations (2.89) and (2.90)

$$\begin{aligned}
r_\| &= \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1} = \frac{\sqrt{7/9} - 1.5\sqrt{1/2}}{\sqrt{7/9} + 1.5\sqrt{1/2}} = -0.0920, \ \rho_\| = 0.0085 \\
r_\perp &= \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{\sqrt{1/2} - 1.5\sqrt{7/9}}{\sqrt{1/2} + 1.5\sqrt{7/9}} = -0.3033, \ \rho_\perp = 0.0920
\end{aligned}$$

$$\boxed{\frac{Q_r}{I_r} = \frac{0.0085 - 0.0920}{0.0085 + 0.0920} = -0.8315, \ \frac{U_r}{I_r} = 0}$$

$$\boxed{\frac{V_r}{I_r} = \pm\frac{2\times 0.0920\times 0.0085}{0.0085 + 0.0920} = \pm 0.5556}$$

Since the perpendicular polarization is much more strongly reflected, the resulting wave is no longer circularly polarized, but to a large degree linearly polarized (in the perpendicular direction).

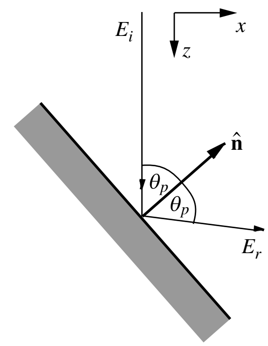
2.5 A circularly polarized wave in air traveling along the z-axis is incident upon a dielectric surface ($n = 1.5$). How must the dielectric-air interface be oriented so that the reflected wave is a linearly polarized wave in the y-z-plane?

Solution

From equations (2.50) through (2.53) it follows that $Q_r/I_r = 1$, $U_r = V_r = 0$ (i.e., linear polarization), if either $E_{r\|}$ or $E_{r\perp}$ vanish. From Fig. 2-9 it follows that $r_\perp \neq 0$ and, therefore $E_{r\perp} \neq 0$ for all incidence directions, while $r_\| = 0$ for $\theta = \theta_p$ (Brewster's angle), or

$$\boxed{\theta_p = \tan^{-1}\frac{n_2}{n_1} = \tan^{-1} 1.5 = 56.31°.}$$

The resulting wave is purely perpendicular-polarized, i.e., $\hat{\mathbf{e}}_\perp$ must lie in the $y-z$ plane, or $\hat{\mathbf{e}}_\|$ must be in the $x-z$ plane. Therefore, the surface may be expressed in terms of its surface normal as $\hat{\mathbf{n}} = \hat{\mathbf{\imath}}\sin\theta_p - \hat{\mathbf{k}}\cos\theta_p = (\hat{\mathbf{\imath}} - 1.5\hat{\mathbf{k}})\big/\sqrt{3.25}$.



2.6 A polished platinum surface is coated with a 1 μm thick layer of MgO.

(a) Determine the material's reflectivity in the vicinity of $\lambda = 2\,\mu$m (for platinum at 2 μm $m_{\rm Pt} = 5.29 - 6.71\,i$, for MgO $m_{\rm MgO} = 1.65 - 0.0001\,i$).

(b) Estimate the thickness of MgO required to reduce the average reflectivity in the vicinity of 2 μm to 0.4. What happens to the interference effects for this case?

Solution

(a) The desired overall reflectivity must be calculated from equation (2.124) after determining the relevant reflection coefficient. From equation (2.122)

$$\widetilde{r}_{12} = \frac{1-m_2}{1+m_2} \simeq \frac{1-n_2}{1+n_2} = \frac{1-1.65}{1+1.65} = -0.2453$$

since $k_2 \ll 1$, and $r_{12} = 0.2453$. $\widetilde{r}_{23}$ may also be calculated from equation (2.122) or, more conveniently, from equation (2.126):

$$r_{23}^2 = \frac{(1.65-5.29)^2+6.71^2}{(1.65+5.29)^2+6.71^2} = 0.6253 \text{ or } r_{23} = 0.7908.$$

Since the real part of $\widetilde{r}_{12} < 0$ it follows that $\delta_{12} = \pi$, while

$$\tan\delta_{23} = \frac{2(1.65{\times}6.71 - 5.29{\times}10^{-4})}{1.65^2 + 10^{-8} - (5.29^2{+}6.71^2)} = -0.3150.$$

Since the $\Im(r_{23}) > 0$ (numerator) and $\Re(r_{23}) < 0$ (denominator) δ_{23} lies in the second quadrant, $\pi/2 < \delta_{23} < \pi$, or $\delta_{23} = 2.8364$. Also $\zeta_{12} = 4\pi \times 1.65 \times 1\,\mu\text{m}/2\,\mu\text{m} = 10.3673$, and

$$\cos\left[\delta_{12} \pm (\delta_{23} - \zeta_{12})\right] = \cos\left[\pi \pm (2.8364 - 10.3673)\right] = -0.3175.$$

Also $\kappa_2 d = 4\pi \times 10^{-4} \times 1\,\mu\text{m}/2\,\mu\text{m} = 2\pi \times 10^{-4}$ and $\tau = e^{-\kappa_2 d} = 0.9994 \simeq 1$. Thus

$$R = \frac{0.2453^2 + 2{\times}0.2453{\times}0.7908{\times}(-0.3175) + 0.7908^2}{1 + 2{\times}0.2453{\times}0.7908{\times}(-0.3175) + 0.2453^2{\times}0.7908^2}$$

$$\boxed{R = 0.6149.}$$

(b) The cos in the numerator fluctuates between $-1 < \cos < +1$. The average value for R is obtained by dropping the cos-term. Then

$$R_{\rm av} = \frac{r_{12}^2 + r_{23}^2\tau^2}{1 + r_{12}^2 r_{23}^2 \tau^2},$$

or

$$\tau^2 = \frac{R_{\rm av} - r_{12}^2}{r_{23}^2(1-r_{12}^2)} = \frac{0.4 - 0.2453^2}{0.7908^2(1-0.2453^2)} = 0.5782,$$

$$d = -\frac{1}{\kappa_2}\ln\tau = -\frac{1}{2\kappa_2}\ln\tau^2 = \frac{-\ln 0.5782}{4\pi\times10^{-4}\,\mu\text{m}^{-1}} = 43.6\,\mu\text{m}.$$

More accurate is the averaged expression, equation (2.129)

$$R_{\rm av} = \rho_{12} + \frac{\rho_{23}(1-\rho_{12})^2\tau^2}{1-\rho_{12}\rho_{23}\tau^2}$$

or

$$\tau^2 = \frac{R_{\rm av} - \rho_{12}}{\rho_{23}\left[(R_{\rm av}-\rho_{12})\rho_{12} + (1-\rho_{12})^2\right]} = \frac{R_{\rm av}-\rho_{12}}{\rho_{23}\left[1-(2-R_{\rm av})\rho_{12}\right]}$$

$$= \frac{0.4 - 0.2453^2}{0.7908^2\left[1 - 1.6\times0.2453^2\right]} = 0.6013$$

and

$$\boxed{d = \frac{-\ln 0.6013}{4\pi \times 10^{-4}\,\mu\text{m}^{-1}} = 40.47\,\mu\text{m}}$$

For such a large d, it follows that $\zeta_2 \simeq 40 \times 10.3673 \simeq 450$. A full interference period is traversed if $\zeta_2 \simeq 450 \pm \pi$. Around $\lambda = 2\,\mu$m this implies a full period is traversed between $2\,\mu$m $\pm$ $0.014\,\mu$m. Such interference effects will rarely be observed because *(i)* the detector will not respond to such small wavelength changes, and *(ii)* the slightest inaccuracies in layer thickness will eliminate the interference effects.

Note: since incoming radiation at $\lambda_0 = 2\,\mu$m has a wavelength of $\lambda = \lambda_0/n_1 = 2/1.65 = 1.2.\,\mu$m, m_{Pt} should really be evaluated at $1.21\,\mu$m.