Solutions - Problems in Statistics

Section 1 Measures of Center and Variation

1. (a) The mean is

$$\overline{x} = \frac{-4-6+2.5+3+8+5-3-4+8+3+2.2}{11} = 1.3364.$$

For the median, write the list in nondecreasing order:

$$-6, -4, -4, -3, 2.2, 2.5, 3, 3, 5, 8, 8.$$

With an odd number $N = 11$ of numbers in the list, the median is the sixth number, 2.5. Here, 6 is obtained as

$$\frac{N+1}{2} = \frac{12}{2} = 6.$$

There are five numbers of the list to the left of 2.5, and five numbers of the set to the right.

The standard deviation s is

$$s = \sqrt{\frac{239.44}{10}} = 4.8933.$$

(b) The mean is

$$\overline{x} = \frac{1+1+1-1+2+3-1+4+2}{9} = 1.3333.$$

Arranged in nondecreasing order, the sequence is

$$-1, -1, 1, 1, 1, 2, 2, 3, 4.$$

With $N = 9$ numbers, the median is the fifth number from the left. This is 1 (not just any 1, but the third of three ones, from the left). There are four numbers of the set to the left of this 1, and four numbers to the right.

The standard deviation is

$$s = \sqrt{\frac{22}{8}} = 1.6583.$$

(c) The mean is

$$\overline{x} = \frac{3-4+2+1.5-4-4+2+1+7}{9} = 0.5.$$

In nondecreasing order, the list is

$$-4, -4, -4, 1, 1.5, 2, 2, 3, 7.$$

The median is 1.5.

The standard deviation is

$$s = \sqrt{\frac{115}{8}} = 3.7914.$$

(d) The mean is

$$\overline{x} = \frac{9.3 + 9.5 + 9.7 + 10 + 8.4 + 8.7 + 8.8 + 8.8 + 4.1}{9} = 8.5889.$$

As a nondecreasing list, we have

$$4.1, 8.4, 8.7, 8.8, 8.8, 9.3, 9.5, 9.7, 10.$$

The median is 8.8. This is the fifth number from the left in the list, and is the second 8.8 from the left.

The standard deviation is

$$s = \sqrt{\frac{24,849}{8}} = 1.7624.$$

(e) The mean is

$$\overline{x} = \frac{-16 - 14 - 10 + 0 + 0 + 1 + 1 + 3 + 5 + 7}{10} = -2.3.$$

The data is already listed in nondecreasing order. the median is the average of the fifth and sixth numbers in this list. This median is

$$\frac{0+1}{2} = \frac{1}{2}.$$

In this case that the number of data points is even, the median need not be a number in the set.

The standard deviation is

$$s = \sqrt{\frac{584.1}{9}} = 8.0561.$$

2. Making use of the frequency table, compute the mean as

$$\overline{x} = \frac{4(-12) + 2(-9.7) + 6(-8) + 4(-7.6) + 12(-5.1) + 3(4)}{31} = -6.2903.$$

The data is given in nondecreasing order. The number of entries is an odd number, 31, so the median is the sixteenth number from the left, -7.6 (the right-most -7.6 in the list). This is obtained by using the frequencies to count up to the sixteenth number from the left.

The standard deviation is

$$s = \sqrt{\frac{512.73}{30}} = 4.1341.$$

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Table 1: Sums of pairs of dice, Problem 1, Section 2.

3. The mean is

$$\overline{x} = \frac{4(-3) + 2(-1) + 6(0) + 4(1) + 12(3) + 3(4)}{31} = 1.2258.$$

The median is the sixteenth number from the left, or 1 (the last 1 to the right in the ordered list).

The standard deviation is

$$s = \sqrt{\frac{151.42}{30}} = 2.2466.$$

Section 2 Random Variables and Probability Distributions

1. If we roll two dice, there are thirty-six possible outcomes. The sums of the numbers that can come up on the two dice are listed in Table 1.

   For example, if o is the outcome that one die comes up 2 and the other 3, then the sum of the dice is 5, so $X(o) = 5$.

   The table gives all of the values that $X(o)$ can take on, over all outcomes o of the experiment. Each value is listed as often as it occurs as a value of X. For example, 4 occurs three times, because $X(o) = 4$ for three different outcomes (namely $(2,2)$, $(1,3)$ and $(3,1)$).

   Define a probability distribution P on X by by letting $P(x)$ be the probability of x, for each value x that X can assume.

   For example, since 2 occurs once out of 36 entries in this table, assign to this value of X the probability

$$P(2) = \frac{1}{36}.$$

   Similarly, 11 occurs twice, so give the value 11 of X the probability

$$P(11) = \frac{2}{36}.$$

Since 3 occurs twice in the table, $P(3) = 2/36$, and so on.

Calculating $P(n)$ for each n in the table, we obtain

$$P(2) = P(12) = \frac{1}{36}, P(3) = P(11) = \frac{2}{36},$$
$$P(4) = P(10) = \frac{3}{36}, P(5) = P(9) = \frac{4}{36},$$
$$P(6) = P(8) = \frac{5}{36}, P(7) = \frac{6}{36}.$$

Notice that $\sum_x P(x) = 1$, as required for a probability function.

The mean of X is

$$\begin{aligned}
\mu &= \sum_x xP(x) \\
&= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) \\
&+ 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) \\
&= 7.
\end{aligned}$$

This is interpreted to mean that, on average, we expect to come up with a seven if we roll two dice. This is a reasonable expectation in view of the fact that there are more ways to roll 7 than any other sum with two dice.

The standard deviation is

$$\sigma = \sqrt{\sum_x (x-7)^2 P(x)}.$$

To compute this, first compute

$$\begin{aligned}
&\sum_x (x-7)^2 P(x) \\
&= (2-7)^2\left(\frac{1}{36}\right) + (3-7)^2\left(\frac{2}{36}\right) + (4-7)^2\left(\frac{3}{36}\right) \\
&+ (5-7)^2\left(\frac{4}{36}\right) + (6-7)^2\left(\frac{5}{36}\right) + (7-7)^2\left(\frac{6}{36}\right) \\
&+ (8-7)^2\left(\frac{5}{36}\right) + (9-7)^2\left(\frac{4}{36}\right) + (10-7)^2\left(\frac{3}{36}\right) \\
&+ (11-7)^2\left(\frac{2}{36}\right) + (12-7)^2\left(\frac{1}{36}\right) \\
&= 5.8333.
\end{aligned}$$

Then

$$\sigma = \sqrt{5.8333} = 2.4152.$$

2. Flip four coins, with sixteen possible outcomes. If o is an outcome, $X(o)$ can have only two values, namely 1 if two, three, or four tails are in o, or 3 otherwise (one tail or no tails in o). There are five outcomes with one tail or no tail, and eleven with two or more tails, so

$$P(1) = \frac{11}{16} \text{ and } P(3) = \frac{5}{16}.$$

The mean is

$$\mu = \sum_x xP(x) = 1\left(\frac{11}{16}\right) + 3\left(\frac{5}{16}\right) = \frac{26}{16} = 1.625.$$

For the standard deviation of X, compute

$$\begin{aligned}
&\sum_x (x-\mu)^2 P(x) \\
&= (1-1.625)^2\left(\frac{11}{16}\right) + (3-1.625)^2\left(\frac{5}{16}\right) \\
&= 0.85938.
\end{aligned}$$

Then

$$\sigma = \sqrt{0.85938} = 0.92703.$$

3. We have

$$\begin{aligned}
&X(1) = 0, \\
&X(2) = X(3) = X(5) = X(7) = X(11) = X(13) = X(17) = X(19) = 1, \\
&X(4) = X(6) = X(9) = X(10) = X(14) = X(15) = 2, \\
&X(8) = X(12) = X(18) = X(20) = 3, \\
&X(16) = 4.
\end{aligned}$$

The values assumed by X are $0, 1, 2, 3, 4$. From the list of values, we get

$$P(0) = \frac{1}{20}, P(1) = \frac{8}{20}, P(2) = \frac{6}{20}, P(3) = \frac{4}{20}, P(4) = \frac{1}{20}.$$

These are the probabilities of the values of the random variable X.

The mean of X is

$$\begin{aligned}
\mu &= \sum_x xP(x) \\
&= 0\left(\frac{1}{20}\right) + 1\left(\frac{8}{20}\right) + 2\left(\frac{6}{20}\right) \\
&+ 3\left(\frac{4}{20}\right) + 4\left(\frac{1}{20}\right) \\
&= 1.8.
\end{aligned}$$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Table 2: Outcomes of rolling two dice, Problem 4, Section 2.

For the standard deviation of X, first compute

$$\begin{aligned}&\sum_x (x-\mu)^2 P(x)\\&= (0-1.8)^2\left(\frac{1}{20}\right) + (1-1.8)^2\left(\frac{8}{20}\right)\\&+ (2-1.8)^2\left(\frac{6}{20}\right) + (3-1.8)^2\left(\frac{4}{20}\right)\\&+ (4-1.8)^2\left(\frac{1}{20}\right) = 0.9600.\end{aligned}$$

Then

$$\sigma = \sqrt{(0.9600)} = 0.9798.$$

4. The outcomes of two rolls of the dice are displayed in Table 2.

   Then

$$\begin{aligned}&X(n,n) = n \text{ for } n = 1,2,3,4,5,6,\\&X(1,2) = X(2,1) = X(2,4) = X(4,2) = X(3,6) = X(6,3) = 2,\\&X(1,3) = X(3,1) = X(2,6) = X(6,2) = 3,\\&X(1,4) = X(4,1) = 4, X(1,5) = X(5,1) = 5, X(1,6) = X(6,1) = 6,\\&X(2,3) = X(3,2) = X(4,6) = X(6,4) = 3/2,\\&X(2,5) = X(5,2) = 5/2, X(3,4) = X(4,3) = 4/3,\\&X(5,3) = X(3,5) = 5/3, X(4,5) = X(5,4) = 5/4,\\&X(5,6) = X(6,5) = 6/5.\end{aligned}$$

   Using this list to compute probabilities, we find that

$$\begin{aligned}P(1) &= \frac{1}{36}, P(2) = \frac{7}{36}, P(3) = \frac{5}{36},\\P(4) &= \frac{3}{36}, P(5) = \frac{3}{36}, P(6) = \frac{3}{36}.\end{aligned}$$

The mean of X is

$$
\begin{aligned}
\mu &= \sum_x P(x) \\
&= 1\left(\frac{1}{36}\right) + 2\left(\frac{7}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{3}{36}\right) + 6\left(\frac{3}{36}\right) \\
&+ \frac{3}{2}\left(\frac{4}{36}\right) + \frac{5}{2}\left(\frac{2}{36}\right) + \frac{4}{3}\left(\frac{2}{36}\right) + \frac{5}{3}\left(\frac{2}{36}\right) + \frac{5}{4}\left(\frac{2}{36}\right) + \frac{6}{5}\left(\frac{2}{36}\right) \\
&= 2.6916.
\end{aligned}
$$

Using μ, we can compute the standard deviation of X:

$$
\begin{aligned}
&\sum_x (x-\mu)^2 P(x) \\
&= (1-2.6916)^2\left(\frac{1}{36}\right) + (2-2.6916)^2\left(\frac{7}{36}\right) + (3-2.6916)^2\left(\frac{5}{36}\right) \\
&+ (4-2.6916)^2\left(\frac{3}{36}\right) + (5-2.6916)^2\left(\frac{3}{36}\right) + (6-2.6916)^2\left(\frac{3}{36}\right) \\
&+ (3/2-2.6916)^2\left(\frac{4}{36}\right) + (5/2-2.6916)^2\left(\frac{2}{36}\right) + (4/3-2.6916)^2\left(\frac{2}{36}\right) \\
&+ (5/3-2.6916)^2\left(\frac{2}{36}\right) + (5/4-2.6916)^2\left(\frac{2}{36}\right) + (6/5-2.6916)^2\left(\frac{21}{36}\right) \\
&= 2.2442.
\end{aligned}
$$

Then

$$\sigma = \sqrt{2.2442} = 1.4981.$$

5. Draw cards from a (fifty-two card) deck. There are ${}_{52}C_2$ ways to do this, disregarding order. If o is an outcome in which both cards are numbered, then $X(o)$ equals the sum of the numbers on the cards. If exactly one of the cards is a face card or ace, then $X(o) = 11$, and if both cards are chosen from the face cards or aces, then $X(o) = 12$. Therefore the values of $X(o)$ for all possible outcomes are $4, 5, \cdots, 20$. By a routine but tedious counting of the ways the numbered cards can take on various possible totals, we obtain

$$
\begin{aligned}
P(4) &= \frac{6}{1326}, P(5) = \frac{16}{1326}, P(6) = \frac{22}{1326}, P(7) = \frac{32}{1326}, \\
P(8) &= \frac{38}{1326}, P(9) = \frac{48}{1326}, P(10) = \frac{54}{1326}, P(11) = \frac{640}{1326}, \\
P(12) &= \frac{190}{1326}, P(13) = \frac{64}{1326}, P(14) = \frac{54}{1326}, P(15) = \frac{48}{1326}, \\
P(16) &= \frac{38}{1326}, P(17) = \frac{32}{1326}, P(18) = \frac{22}{1326}, P(19) = \frac{16}{1326}, \\
P(20) &= \frac{6}{1326}.
\end{aligned}
$$

Using these, compute the mean of X:

$$\mu = \sum_x xP(x) = 11.566$$

and the standard deviation of X:

$$\sigma = \sqrt{\sum_x (x-\mu)^2 P(x)} = \sqrt{5.4289} = 2.33.$$

6. X takes on values $1, 2, \pi$. In particular,

$$\begin{aligned} X(1) &= X(5) = X(7) = X(11) = X(13) = X(17) \\ &= X(19) = X(23) = X(25) = X(29) = \pi, \\ X(2) &= X(4) = \cdots = X(\text{even integer}) = 1, \\ X(3) &= X(9) = X(15) = X(21) = X(27) = 2. \end{aligned}$$

These enable us to write the probability distribution

$$P(1) = \frac{15}{30} = \frac{1}{2}, P(2) = \frac{5}{30} = \frac{1}{6}, P(\pi) = \frac{10}{30} = \frac{1}{3}.$$

The mean of X is

$$\mu = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{6}\right) + \pi\left(\frac{1}{3}\right) = \frac{5}{6} + \frac{\pi}{3},$$

and this is approximately 1.8805.

The standard deviation of X is the square root of

$$\begin{aligned} &\sum_x (x-\mu)^2 P(x) \\ &= (1-1.8805)^2\left(\frac{1}{2}\right) + (2-1.8805)^2\left(\frac{1}{6}\right) + (\pi-1.8805)^2\left(\frac{1}{3}\right), \end{aligned}$$

which is approximately 0.92014. Then

$$\sigma = \sqrt{92014} = 0.95924.$$

Section 3 The Binomial and Poisson Distributions

1. (a)

$$P(2) = \binom{8}{2}(0.43)^2(1-0.43)^6 = 0.17756.$$

(b)

$$P(3) = \binom{4}{3}(0.7)^3(1-0.7) = 0.4116.$$

(c)

$$P(3) = \binom{6}{3}(0.5)^3(1-0.5)^3 = 0.3125.$$

(d)

$$\begin{aligned}
&P(2)+P(3)+P(4)+P(5) \\
&= \binom{10}{2}(0.6)^2(0.4)^8 + \binom{10}{3}(0.6)^3(0.4)^7 \\
&+ \binom{10}{4}(0.6)^4(0.4)^6 + \binom{10}{5}(0.6)^5(0.4)^5 \\
&= 0.36521.
\end{aligned}$$

(e)

$$\begin{aligned}
P(7)+P(8) &= \binom{8}{7}(0.4)^7(0.6) + \binom{8}{8}(0.4)^8 \\
&= 0.0085197.
\end{aligned}$$

(f)

$$\begin{aligned}
&P(2)+P(3)+P(4) \\
&= \binom{10}{2}(0.58)^2(0.42)^8 + \binom{10}{3}(0.58)^3(0.42)^7 + \binom{10}{4}(0.58)^4(0.42)^6 \\
&= 0.19908.
\end{aligned}$$

(g)

$$\begin{aligned}
P(3)+P(7) &= \binom{10}{3}(0.35)^3(0.65)^7 + \binom{10}{7}(0.35)^7(0.65)^3 \\
&= 0.27342.
\end{aligned}$$

(h)

$$\begin{aligned}
&P(1)+P(3)+P(5) \\
&= \binom{7}{1}(0.24)(0.76)^6 + \binom{7}{3}(0.24)^3(0.76)^4 + \binom{7}{5}(0.24)^5(0.76)^2 \\
&= 0.49481.
\end{aligned}$$